Greedy Flipping of Pancakes and Burnt Pancakes

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Abstract

We prove that a stack of n pancakes is rearranged in all $n!$ ways by repeatedly applying the following rule: *Flip the maximum number of pancakes that gives a new stack*. This complements the previously known pancake flipping Gray code (S. Zaks, *A New Algorithm for Generation of Permutations* BIT 24 (1984), 196–204) which we also describe as a greedy algorithm: *Flip the minimum number of pancakes that gives a new stack*. Surprisingly, these maximum and minimum flip algorithms also rearrange stacks of n 'burnt' pancakes in all $2^n n!$ ways. We conjecture that these four algorithms are essentially the only greedy algorithms for rearranging pancakes and burnt pancakes in all possible ways using flips.

Keywords: greedy algorithm, Gray code, permutations, signed permutations, prefix-reversal, symmetric group, signed symmetric group, Cayley graph, Hamilton cycle

1. Introduction

Take a stack of n distinct pancakes, numbered $1, 2, \ldots, n$ by increasing diameter, and repeat the following: *Flip the maximum number of topmost pancakes that gives a new stack*. For example, if the first stack is 12345 when read from top to bottom, then the second stack is created by flipping all five pancakes to give 54321. To create the third stack from the second stack, we cannot flip all five pancakes (since it would recreate 12345), however we can flip the top four pancakes to give 23451. This process is a greedy algorithm, and Figure 1 illustrates the resulting list of stacks.

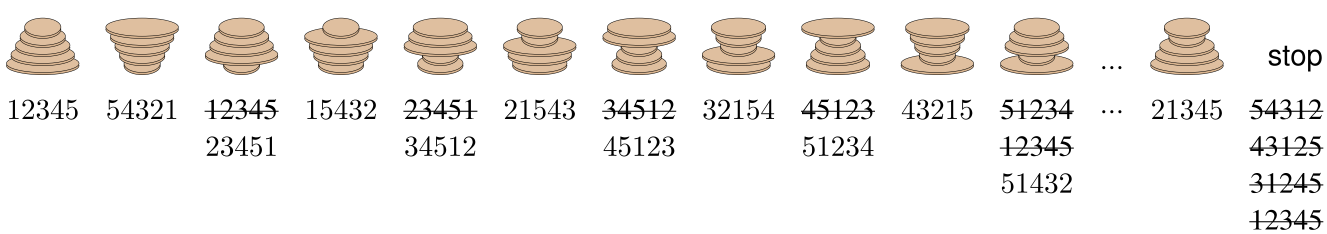

Figure 1: Greedily flipping the maximum number of topmost pancakes from 12345. The order is read from left-to-right, and previously created stacks that are rejected by the algorithm are crossed out. All $5! = 120$ stacks are created. The last stack is 21345 since each flip gives a previous stack. In particular, flipping the top two pancakes gives the first stack 12345.

Formally, each stack of pancakes is a permutation of $\{1, 2, \ldots, n\}$ in one-line notation, and flipping the topmost k pancakes corresponds to a *prefix-reversal* of length k in the permutation. When using the pancake flipping metaphor, the reader can visualize a spatula being used for each flip. The same metaphor can be applied to 'burnt' pancakes that have two distinct sides; the 'burnt' side of each pancake alternates

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facing up and down when it is flipped. In this case, a stack of burnt pancakes is formalized as a *signed permutation* of $\{1, 2, \ldots, n\}$ and flipping the topmost k pancakes corresponds to a *complemented prefix-reversal* of length k in the signed permutation. Overlines are used to represent negative elements in a signed permutation. For example, applying a complemented prefix-reversal of length three to the signed permutation $\bar{1}3\bar{4}\bar{2}5$ results in $4\bar{3}1\bar{2}5$ The greedy algorithm that flips the maximum number of pancakes can also be applied to stacks of burnt pancakes, as illustrated by Figure 2.

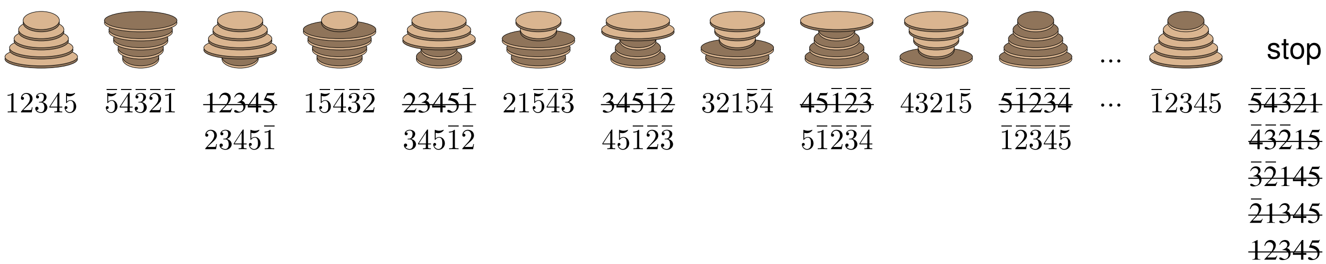

Figure 2: Greedily flipping the maximum number of topmost burnt pancakes starting from 12345. All $2^5 5! = 3840$ stacks are created. The last stack is $\bar{1}2345$ since each flip gives a previous stack. In particular, flipping the top pancake gives the first stack 12345.

Amazingly, the lists generated by the greedy algorithm are both exhaustive for $n = 5$. In other words, the greedy algorithm generates all $5! = 120$ permutations and all $2^5 5! = 3840$ signed permutations before it terminates. We will prove that this result holds for all $n \geq 1$. Furthermore, we prove that the analogous *minimum flip* greedy algorithm also creates all $n!$ permutations and $2^n n!$ signed permutations. Collectively, these four results form the basis for this article.

To understand the significance of these results, let us consider two similar greedy algorithms. A *prefix-rotation of length* j moves the j-th symbol to the beginning and the first $j-1$ symbols are moved one position to the right. For example, 54321 becomes 25431 after a prefix-rotation of length four. A metaphor for this scenario is a vertical column of n distinct balls, where prefix-rotations of length j are performed by grabbing the jth ball and dropping it at the top of the column. Figure 3 shows the result of greedily rotating the maximum length prefix of the permutation representing each container starting from 1234. Similarly, Figure 4 shows the result of greedily rotating the minimum length prefix starting from 1234. In both cases the algorithm terminates before all $4! = 24$ permutations are created.

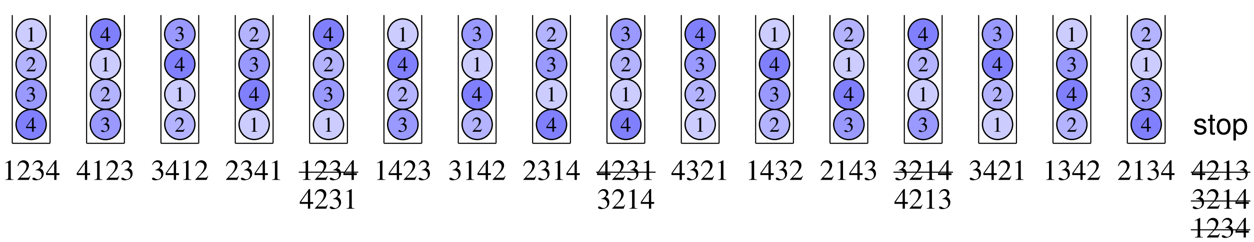

Figure 3: Greedily rotating the maximum length prefix starting from 1234 terminates after creating only 16 permutations.

Readers are likely familiar with the *binary reflected Gray code* [6], which orders the 2^n n-bit binary strings so that successive strings differ by a single bit complementation. In general, the term *Gray code* can be used for any exhaustive ordering of a set of combinatorial objects in which successive objects are "close to each other" according to some measure or operation. For surveys on Gray codes of permutations and other objects see Sedgewick [14], Savage [11], and Section 7.2.1.2 of Knuth [10]. We describe our main results using the language of Gray codes as follows:

(1) The minimum-flip (prefix-reversal) greedy algorithm for permutations produces a Gray code for permutations, and its average flip length approaches e.

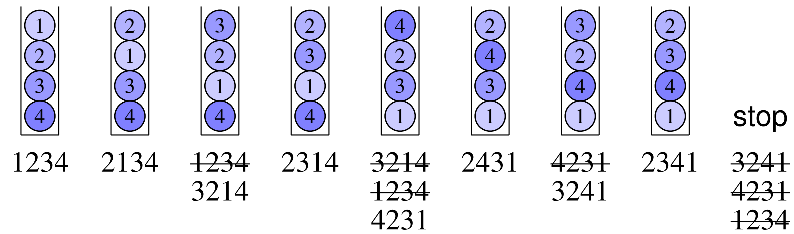

Figure 4: Greedily rotating the minimum length prefix starting from 1234 terminates after creating only 8 permutations.

(2) The minimum-flip (complemented prefix-reversal) greedy algorithm for signed permutations produces a Gray code for signed permutations, and its average flip length approaches $\sqrt{e}$.

(3) The maximum-flip (prefix-reversal) greedy algorithm for permutations produces a Gray code for permutations, and its average flip length approaches $n - \frac{1}{2}$.

(4) The maximum-flip (complemented prefix-reversal) greedy algorithm for signed permutations produces a Gray code for signed permutations, and its average flip length approaches $n - \frac{1}{2}$.

We will see that all four Gray codes are also *cyclic*, meaning that the last and first (signed) permutations differ by a single (complemented) prefix-reversal. To prove these results we derive equivalent recursive formulations for the orders of (signed) permutations and their flip length sequences.

The recursive formulation of the minimum-flip order for permutations, its sequence of flip lengths, and its average flip length was previously published by Zaks [20]. Similarly, the recursive formulation of the minimum-flip order for signed permuations and its sequence of flip lengths was observed by Suzuki, N. Sawada, and Kaneko [17]. The maximum-flip order for permutations was the subject of an extended abstract at LAGOS 2013 [19]. This article contributes the maximum-flip order for signed permutations, the unified greedy interpretation that includes the minimum-flip orders for permutations and signed permutations, and the remaining average flip length analyses. In follow-up articles the authors have used the recursive formulations as the basis for efficient algorithms that generate, rank, and unrank all four orders [12] and for successor rules that determine each successive flip directly from the current stack of (burnt) pancakes [13].

This article serves one additional purpose beyond the above results. Although greedy algorithms have been applied widely across many problem domains, the basic approach has not received significant attention in the area of Gray codes. Recently, a survey was published on this topic [18] and its focus was on greedy reinterpretations of classic Gray codes. This article represents the first case where the greedy method is explicitly used as a starting point, and all variations of the greedy algorithm are considered. This methodical investigation led to discovery of the maximum-flip Gray codes, and to the following uniqueness conjecture: Algorithms (1) through (4) are the only greedy flip Gray codes for permutations and signed permutations when n is sufficiently large.

The authors chose pancake flipping as the setting for the first thorough investigation of greedy Gray codes due to the nice structure of the resulting orders, and the interesting mathematical and computational history involved with flipping pancakes. Pancake flipping was initially examined in the context of sorting [3], with bounds [5, 2], algorithms [9], and complexity results [1] attracting wider media attention [16]. The metaphor has also been useful in combinatorial genetics [4] and bacterial computing [7], and the underlying pancake and burnt pancake Cayley graphs are used as interconnection networks [15, 8]. In some sense our pancake Gray codes are opposite to pancake sorting since they involve rearranging the sorted stack into all possible stacks, as opposed to rearranging all possible stacks into the sorted stack.

In the following section, we outline the notation used in the remainder of this paper. In Section 3 we formalize the greedy approach for combinatorial generation described in [18]. We provide a detailed examination of the minimum-flip orders for permutations (Section 4) and signed permutations (Section 5), and then we provide a detailed examination of the maximum-flip orders for permutations (Section 6) and signed permutations (Section 7). We conclude in Section 8 with open problems and avenues for future research, including our uniqueness conjectures.

2. Notation

In this paper we are concerned with providing Gray code listings for the permutations and signed permutations for the set $\mathbf{S} = \{1, 2, 3, \ldots, n\}$. However, for the proofs we must consider arbitrary sets of n elements. For example the six permutations of $\mathbf{S} = \{1, 2, 4\}$ are $\{124, 142, 214, 241, 412, 421\}$ and the eight signed permutations of $\mathbf{S} = \{1, 4\}$ are $\{14, 41, \bar{1}4, 4\bar{1}, 1\bar{4}, \bar{4}1, \bar{1}\bar{4}, \bar{4}\bar{1}\}$ where $\bar{p}$ denotes $-p$.

Let the set of (unsigned) permutations of an n-set be denoted by $\mathbb{P}(n)$. Given $\mathbf{p} = p_1p_2p_3 \cdots p_n \in \mathbb{P}(n)$, we will use the following notation for $1 \leq j \leq n$:

- $\mathsf{flip}_j(\mathbf{p}) = p_jp_{j-1} \cdots p_1p_{j+1} \cdots p_n$ denotes a flip (prefix-reversal) of length j, and
- $\mathbf{p}(j) = p_{j+1} \cdots p_np_1 \cdots p_{j-1}$ denotes a full rotation to the left by j positions followed by the removal of the element p_j.

Let the set of signed permutations of an n-set be denoted by $\overline{\mathbb{P}}(n)$. Given $\mathbf{p} = p_1p_2p_3 \cdots p_n \in \overline{\mathbb{P}}(n)$, we will use the following notation for $1 \leq j \leq n$:

- $\overline{\mathsf{flip}}_j(\mathbf{p}) = \bar{p}_j\bar{p}_{j-1} \cdots \bar{p}_1p_{j+1} \cdots p_n$ denotes a flip (complemented prefix reversal) of length j,
- $\mathsf{flipSign}(\mathbf{p}) = \bar{p}_1\bar{p}_2\bar{p}_3 \cdots \bar{p}_n$ flips the sign of every element,
- $\mathbf{p}'(j) = \bar{p}_{j+1} \cdots \bar{p}_np_1p_2 \cdots p_{j-1}$, and
- $\bar{\mathbf{p}}(j) = p_{j+1} \cdots p_n\bar{p}_1\bar{p}_2 \cdots \bar{p}_{j-1} = \mathsf{flipSign}(\mathbf{p}'(i))$.

For both signed and unsigned permutations we will use the following notation for a permutation $\mathbf{p}$:

- $\mathbf{p} \cdot n$ denotes the concatenation of the symbol n to the permutation $\mathbf{p}$.

Consider a sequence of unsigned permutations $\rho = \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ and an integer sequence $\phi = f_1, f_2, \ldots f_{m-1}$ for some $m > 1$. We say that ϕ is the *flip-sequence* for ρ if $\mathbf{p}_{i+1} = \mathsf{flip}_{f_i}(\mathbf{p}_i)$ for $1 \leq i \leq m-1$. Similarly, if $\rho = \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ is a sequence of m signed permutations then $\phi = f_1, f_2, \ldots f_{m-1}$ is said to be the *flip-sequence* for ρ if $\mathbf{p}_{i+1} = \overline{\mathsf{flip}}_{f_i}(\mathbf{p}_i)$ for $1 \leq i \leq m-1$.

When describing sequences, we let x^k denote k repeated concatenations of the sequence x. For example $(1, 3)^4 = 1, 3, 1, 3, 1, 3, 1, 3$.

3. Greedy Approach

In this section we outline the greedy approach discussed in [18]. The approach is applied to a set $\mathcal{S}$ of combinatorial objects, an initial object $s \in \mathcal{S}$, and a prioritized list of operations $ops = \langle \mathsf{op}_1, \mathsf{op}_2, \mathsf{op}_3, \ldots, \mathsf{op}_m \rangle$ where $\mathsf{op}_i : \mathcal{S} \to \mathcal{S}$ for all $1 \leq i \leq m$. Algorithm 1 produces a (not necessarily exhaustive) list $\mathcal{L}$ of elements from $\mathcal{S}$. The function GREEDYCHOICE returns the smallest integer j such that $\mathsf{op}_j(s)$ is in $\mathcal{S}$ but is not already in $\mathcal{L}$; the function returns 0 if no such operation exists.

This generic greedy algorithm provides a simple unified way for describing many previously constructed Gray codes [18]. Furthermore, it provides a simple "experimental" approach for discovering new Gray codes. The approach begins by experimenting with small input sizes. If experiments are successful for small input sizes, then larger input sizes are considered. If these experiments are successful

Algorithm 1 Greedy approach for listing combinatorial objects in $\mathcal{S}$ starting with s using operations ops

```
procedure GreedyGen
    L ← s
    repeat
        j ← GreedyChoice
        if j ≠ 0 then
            s ← op_j(s)
            append s to L
    until j = 0
```

for larger inputs, then the lists can be examined for patterns and properties that are contributing to the success of the algorithm. The next step is to formally prove that the algorithm works for aribtrarily large input sizes. Once this proof is established, attention can be shifted to developing efficient algorithms for generation, ranking, and unranking. The following list highlights this research strategy for a given set $\mathcal{S}$:

1. Prioritize a sequence of operations $ops = \langle \mathsf{op}_1, \mathsf{op}_2, \mathsf{op}_3, \ldots, \mathsf{op}_m \rangle$.
2. Select an initial $s \in \mathcal{S}$.
3. Run GreedyGen to create listing $\mathcal{L}$.
   (a) If $\mathcal{L}$ is not exhaustive, GOTO 1. and try a new sequence of operations or initial object.
   (b) If $\mathcal{L}$ is exhaustive, continue.
4. Repeat the above steps for sufficiently large inputs.
5. Study the listing and try to prove that it works in general.
6. [Optional] Develop efficient algorithms for ranking, unranking, and generating the Gray code.

We let $\mathsf{Greedy}(\mathcal{S}, s, ops)$ denote the listing $\mathcal{L}$ produced by the greedy approach for a set $\mathcal{S}$, an initial element $s \in \mathcal{S}$, and an ordered sequence of operations ops.

Now we apply the greedy approach to pancake flipping. The objects are (signed) permutations, and the operations are (signed) flips. Observe that the initial permutation is not relevant to the success of the greedy approach for (signed) permutations since the elements can be relabeled. Our first experiments were on $\mathbb{P}(n)$ with an arbitrary initial permutation $\mathbf{p} = p_1p_2p_3 \cdots p_n$. Using the greedy approach with operations $\{\mathsf{flip}_2, \mathsf{flip}_3, \mathsf{flip}_4, \ldots, \mathsf{flip}_n\}$ we were able to verify that both the minimum-flip and the maximum-flip approaches produced exhaustive listings for $1 \leq n \leq 11$. Next we considered $\overline{\mathbb{P}}(n)$ with an arbitrary initial permutation $\mathbf{p} = p_1p_2p_3 \cdots p_n$. Using the greedy approach with operations $\{\overline{\mathsf{flip}}_1, \overline{\mathsf{flip}}_2, \overline{\mathsf{flip}}_3, \ldots, \overline{\mathsf{flip}}_n\}$ we were able to verify that both the minimum-flip and the maximum-flip approaches produced exhaustive listing for $1 \leq n \leq 7$.

In the next four sections, we will prove that each of these greedy approaches produce flip Gray codes for $n \geq 1$ for an arbitrary initial permutation $\mathbf{p} = p_1p_2p_3 \cdots p_n$. To simplify the notation, let:

- $\mathsf{MinGreedy}(\mathbf{p}) \;=\; \mathsf{Greedy}(\,\mathbb{P}(n), \mathbf{p}, \langle \mathsf{flip}_2, \mathsf{flip}_3, \mathsf{flip}_4, \ldots, \mathsf{flip}_n \rangle)$,
- $\mathsf{MaxGreedy}(\mathbf{p}) \;=\; \mathsf{Greedy}(\,\mathbb{P}(n), \mathbf{p}, \langle \mathsf{flip}_n, \mathsf{flip}_{n-1}, \mathsf{flip}_{n-2}, \ldots, \mathsf{flip}_2 \rangle)$,
- $\overline{\mathsf{MinGreedy}}(\mathbf{p}) \;=\; \mathsf{Greedy}(\,\overline{\mathbb{P}}(n), \mathbf{p}, \langle \overline{\mathsf{flip}}_1, \overline{\mathsf{flip}}_2, \overline{\mathsf{flip}}_3, \ldots, \overline{\mathsf{flip}}_n \rangle)$,
- $\overline{\mathsf{MaxGreedy}}(\mathbf{p}) \;=\; \mathsf{Greedy}(\,\overline{\mathbb{P}}(n), \mathbf{p}, \langle \overline{\mathsf{flip}}_n, \overline{\mathsf{flip}}_{n-1}, \overline{\mathsf{flip}}_{n-2}, \ldots, \overline{\mathsf{flip}}_1 \rangle)$.

Experiments and conjectures involving other orderings of the flip operations are discussed in Section 8.

4. Minimum Flips for Permutations

In this section we consider the minimum-flip greedy algorithm for permutations. We will prove that the greedy ordering corresponds to one that was initially discovered by Zaks [20]. One distinction is that we use prefix-reversals as opposed to suffix-reversals. We begin by looking at the greedy listing $\mathsf{MinGreedy}(1234)$. The length of the flip to go from one permutation to the next is given in parentheses after each permutation.

Example 4.1. $\mathsf{MinGreedy}(1234)$ *(read down, then left to right):*

1 2 3 4 (2)	4 1 2 3 (2)	3 4 1 2 (2)	2 3 4 1 (2)
2 1 3 4 (3)	1 4 2 3 (3)	4 3 1 2 (3)	3 2 4 1 (3)
3 1 2 4 (2)	2 4 1 3 (2)	1 3 4 2 (2)	4 2 3 1 (2)
1 3 2 4 (3)	4 2 1 3 (3)	3 1 4 2 (3)	2 4 3 1 (3)
2 3 1 4 (2)	1 2 4 3 (2)	4 1 3 2 (2)	3 4 2 1 (2)
3 2 1 4 (4)	2 1 4 3 (4)	1 4 3 2 (4)	4 3 2 1 (4)

Observe that the permutations at the top of each column are equivalent under rotation and that each column has the same flip-sequence. If we ignore the final flip to return to the initial permutation, then we will prove that the sequence σ_n given by Zaks [20], is in fact the flip-sequence for $\mathsf{MinGreedy}(\mathbf{p})$:

$$\sigma_n = \begin{cases} 2 & \text{if } n = 2 \\ (\sigma_{n-1}, n)^{n-1}, \sigma_{n-1} & \text{if } n > 2. \end{cases}$$

First, we need to further understand the ordering of permutations produced. By studying the greedy listing from the example and using the recurrence σ_n, we can deduce a simple recurrence to list all permutations. Let $\mathbf{p} = p_1p_2p_3 \cdots p_n$ denote a permutation of an arbitrary n-set $\mathbf{S}$. Recall that $\mathbf{p}(i) = p_{i+1} \cdots p_n p_1 \cdots p_{i-1}$ where $1 \le i \le n$. Consider the following sequence $\mathbf{Min}(\mathbf{p})$, defined by the following recurrence:

$$\mathbf{Min}(\mathbf{p}) = \begin{cases} \mathbf{p} & \text{if } n = 1 \\ \mathbf{Min}(\mathbf{p}(n)) \cdot p_n,\ \mathbf{Min}(\mathbf{p}(n-1)) \cdot p_{n-1}, \ldots,\ \mathbf{Min}(\mathbf{p}(1)) \cdot p_1 & \text{if } n \ge 2. \end{cases} \quad (1)$$

As an example: $\mathbf{Min}(1234) = \mathbf{Min}(123) \cdot 4,\ \mathbf{Min}(412) \cdot 3,\ \mathbf{Min}(341) \cdot 2,\ \mathbf{Min}(234) \cdot 1.$

Lemma 4.2. *For $n \ge 1$, the first and last permutations in the listing $\mathbf{Min}(p_1p_2p_3 \cdots p_n)$ are $p_1p_2p_3 \cdots p_n$ and $p_n \cdots p_3p_2p_1$ respectively.*

Proof. By induction. In the base case when $n = 1$, $\mathbf{Min}(p_1) = p_1$. Inductive Hypothesis: For $n \ge 1$, assume that the first and last permutation in $\mathbf{Min}(p_1p_2p_3 \cdots p_n)$ are $p_1p_2p_3 \cdots p_n$ and $p_n \cdots p_3p_2p_1$ respectively. Observe that the first permutation of $\mathbf{Min}(p_1p_2p_3 \cdots p_{n+1})$ is the first permutation of $\mathbf{Min}(p_1p_2p_3 \cdots p_n) \cdot p_{n+1}$. Applying the inductive hypothesis this permutation is $p_1p_2p_3 \cdots p_{n+1}$. Similarly, the last permutation of $\mathbf{Min}(p_1p_2p_3 \cdots p_{n+1})$ is the last permutation of $\mathbf{Min}(p_2p_3 \cdots p_{n+1}) \cdot p_1$. Applying the inductive hypothesis this permutation is $p_{n+1} \cdots p_3p_2p_1$. □

Lemma 4.3. *For $n \ge 2$, the flip-sequence for $\mathbf{Min}(\mathbf{p})$ is σ_n.*

Proof. By induction. In the base case $\mathbf{Min}(p_1p_2) = p_1p_2, p_2p_1$ and the flip-sequence is $\sigma_2 = 2$. Inductive Hypothesis: For $n \ge 2$ assume that the sequence of flips used to create $\mathbf{Min}(p_1p_2 \cdots p_n)$ is given by σ_n. Consider $\mathbf{Min}(\mathbf{p})$ where $\mathbf{p} = p_1p_2p_3 \cdots p_{n+1}$. By definition:

▷ $\mathbf{p}(i) = p_{i+1} \cdots p_{n+1} p_1 \cdots p_{i-1}$ and
▷ $\mathbf{p}(i{-}1) = p_i \cdots p_{n+1} p_1 \cdots p_{i-2}$.

Thus, for $n{+}1 \geq i > 1$, Lemma 4.2 states that the last permutation of $\mathbf{Min}(\mathbf{p}(i)){\cdot}p_i$ is $p_{i-1} \cdots p_1 p_{n+1} \cdots p_i$ and the first permutation of $\mathbf{Min}(\mathbf{p}(i-1)) \cdot p_{i-1}$ is $p_i \cdots p_{n+1} p_1 \cdots p_{i-1}$. These two permutations differ by a flip of length $n+1$. By applying the inductive hypothesis, the flip-sequence for $\mathbf{Min}(\mathbf{p})$ is given by $(\sigma_n, n+1)^n, \sigma_n$ which is exactly σ_{n+1}. □

Theorem 4.4. *For $n \geq 1$, the listing $\mathbf{Min}(\mathbf{p})$ is a flip Gray code for permutations, where the first and last permutations differ by a flip of length n.*

Proof. From Lemma 4.3, the flip-sequence for $\mathbf{Min}(\mathbf{p})$ is given by the sequence σ_n from [20]. It is easy to see that the length of the flip-sequence σ_n is $n! - 1$. Inductively, it is trivial to observe that each permutation of $\mathbf{Min}(\mathbf{p})$ is unique. Thus, each of the $n!$ permutations are listed exactly once, making $\mathbf{Min}(\mathbf{p})$ a permutation flip Gray code. Finally, from Lemma 4.2, the first and last permutation of the listing differ by a flip of length n. □

Lemma 4.5. *For $n \geq 2$, the flip-sequence for* $\mathsf{MinGreedy}(\mathbf{p})$ *is σ_n.*

Proof. By contradiction. Suppose the sequence of flips used by $\mathsf{MinGreedy}(\mathbf{p})$ differs from σ_n. Let j be the smallest value such that the j-th flip used to create $\mathsf{MinGreedy}(\mathbf{p})$ differs from the j-th value of σ_n. Let these flip lengths be s and t respectively. Since $\mathsf{MinGreedy}(\mathbf{p})$ follows a greedy minimum-flip strategy and because σ_n produces a valid permutation Gray code by Theorem 4.4 where no permutations are repeated, it must be that $s < t$. Let $\mathbf{q} = q_1 q_2 q_3 \cdots q_n$ denote the j-th permutation in the listing $\mathsf{MinGreedy}(\mathbf{p})$; it is the permutation prior to the j-th flip. By the recursive definition of σ_n, the $(t-1)!-1$ elements immediately prior to the j-th element in σ_n each have value less than t. This means that $\mathsf{MinGreedy}(\mathbf{p})$ would apply $(t-1)!$ consecutive flips of length less than t. But this contradicts that fact that $\mathsf{MinGreedy}(\mathbf{p})$ produces a list of unique strings. □

Corollary 4.6. *For $n \geq 1$, the listings* $\mathsf{MinGreedy}(\mathbf{p})$ *and* $\mathbf{Min}(\mathbf{p})$ *are equivalent.*

Proof. By definition, $\mathsf{MinGreedy}(\mathbf{p})$ starts with permutation $\mathbf{p}$ and by Lemma 4.2 $\mathbf{Min}(\mathbf{p})$ also starts with $\mathbf{p}$. Since they are created by the same flip-sequence by Lemma 4.3 and Lemma 4.5, they will produce the same listing of permutations. □

As n goes to infinity, Zaks [20] showed that the average flip length at each step approaches $e \approx 2.7182818$ from below. This analysis includes the flip to return to the initial permutation.

5. Minimum Flips for Signed Permutations

The results in this section for signed permutations mirror the results for unsigned permutations. Again, we begin by looking at an example, this time considering the greedy listing $\overline{\mathsf{MinGreedy}}(123)$. The length of the flip to go from one signed permutation to the next is given in parentheses after each signed permutation.

Example 5.1. $\overline{\mathsf{MinGreedy}}(123)$ *(read down, then left to right):*

$1\,2\,3$ (1)	$\bar{3}\,1\,2$ (1)	$\bar{2}\,\bar{3}\,1$ (1)	$\bar{1}\,\bar{2}\,\bar{3}$ (1)	$3\,\bar{1}\,\bar{2}$ (1)	$2\,3\,\bar{1}$ (1)
$\bar{1}\,2\,3$ (2)	$3\,1\,2$ (2)	$2\,\bar{3}\,1$ (2)	$1\,\bar{2}\,\bar{3}$ (2)	$\bar{3}\,\bar{1}\,\bar{2}$ (2)	$\bar{2}\,3\,\bar{1}$ (2)
$\bar{2}\,1\,3$ (1)	$\bar{1}\,\bar{3}\,2$ (1)	$3\,\bar{2}\,1$ (1)	$2\,\bar{1}\,\bar{3}$ (1)	$1\,3\,\bar{2}$ (1)	$\bar{3}\,2\,\bar{1}$ (1)
$2\,1\,3$ (2)	$1\,\bar{3}\,2$ (2)	$\bar{3}\,\bar{2}\,1$ (2)	$\bar{2}\,\bar{1}\,\bar{3}$ (2)	$\bar{1}\,3\,\bar{2}$ (2)	$3\,2\,\bar{1}$ (2)
$\bar{1}\,\bar{2}\,3$ (1)	$3\,\bar{1}\,2$ (1)	$2\,3\,1$ (1)	$1\,2\,\bar{3}$ (1)	$\bar{3}\,1\,\bar{2}$ (1)	$\bar{2}\,\bar{3}\,\bar{1}$ (1)
$1\,\bar{2}\,3$ (2)	$\bar{3}\,\bar{1}\,2$ (2)	$\bar{2}\,3\,1$ (2)	$\bar{1}\,2\,\bar{3}$ (2)	$3\,1\,\bar{2}$ (2)	$2\,\bar{3}\,\bar{1}$ (2)
$2\,\bar{1}\,3$ (1)	$1\,3\,2$ (1)	$\bar{3}\,2\,1$ (1)	$\bar{2}\,1\,\bar{3}$ (1)	$\bar{1}\,\bar{3}\,\bar{2}$ (1)	$3\,\bar{2}\,\bar{1}$ (1)
$\bar{2}\,\bar{1}\,3$ (3)	$\bar{1}\,3\,2$ (3)	$3\,2\,1$ (3)	$2\,1\,\bar{3}$ (3)	$1\,\bar{3}\,\bar{2}$ (3)	$\bar{3}\,\bar{2}\,\bar{1}$ (3)

In each column of this example, note that the last element of each signed permutation is the same. Additionally, each column has the same flip-sequence. If we ignore the final flip to return to the initial signed permutation, then we will prove that the following flip-sequence $\overline{\sigma}_n$ is the same sequence used by $\overline{\mathsf{MinGreedy}}(\mathbf{p})$:

$$\overline{\sigma}_n = \begin{cases} 1 & \text{if } n = 1 \\ (\overline{\sigma}_{n-1}, n)^{2n-1}, \overline{\sigma}_{n-1} & \text{if } n > 1. \end{cases}$$

However, to formally prove that this flip-sequence is the one used by $\overline{\mathsf{MinGreedy}}(\mathbf{p})$, we need to further understand the ordering of signed permutations produced.

Fortunately, by studying the greedy listing from the example and using the recurrence $\overline{\sigma}_n$, we can deduce a simple recurrence to list all signed permutations. Let $\mathbf{p} = p_1p_2p_3 \cdots p_n$ denote a signed permutation of an arbitrary n-set $\mathbf{S}$. Recall the notation from Section 2 for $1 \leq i \leq n$:

- $\mathbf{p}'(i) = \bar{p}_{i+1} \cdots \bar{p}_n p_1 p_2 \cdots p_{i-1}$, and
- $\bar{\mathbf{p}}(i) = \mathsf{flipSign}(\mathbf{p}'(i)) p_{i+1} \cdots p_n \bar{p}_1 \bar{p}_2 \cdots \bar{p}_{i-1}$.

We will show that following recurrence for $\overline{\mathbf{Min}}(\mathbf{p})$ produces the same listing as $\overline{\mathsf{MinGreedy}}(n)$:

$$\overline{\mathbf{Min}}(\mathbf{p}) = \begin{cases} p_1, \bar{p}_1 & \text{if } n = 1 \\ \\ \overline{\mathbf{Min}}(\mathbf{p}'(n)) \cdot p_n,\ \overline{\mathbf{Min}}(\mathbf{p}'(n-1)) \cdot p_{n-1}, \ldots,\ \overline{\mathbf{Min}}(\mathbf{p}'(1)) \cdot p_1, & \text{if } n \geq 2. \\ \overline{\mathbf{Min}}(\bar{\mathbf{p}}(n)) \cdot \bar{p}_n,\ \overline{\mathbf{Min}}(\bar{\mathbf{p}}(n-1)) \cdot \bar{p}_{n-1}, \ldots,\ \overline{\mathbf{Min}}(\bar{\mathbf{p}}(1)) \cdot \bar{p}_1 & \end{cases} \tag{2}$$

As an example:

$$\overline{\mathbf{Min}}(123) = \overline{\mathbf{Min}}(12) \cdot 3,\ \overline{\mathbf{Min}}(\bar{3}1) \cdot 2,\ \overline{\mathbf{Min}}(\bar{2}\bar{3}) \cdot 1,\ \overline{\mathbf{Min}}(\bar{1}\bar{2}) \cdot \bar{3},\ \overline{\mathbf{Min}}(3\bar{1}) \cdot \bar{2},\ \overline{\mathbf{Min}}(23) \cdot \bar{1}.$$

Lemma 5.2. *For $n \geq 1$, the first and last signed permutations in the listing $\overline{\mathbf{Min}}(p_1p_2p_3 \cdots p_n)$ are $p_1p_2p_3 \cdots p_n$ and $\bar{p}_n \cdots \bar{p}_3\bar{p}_2\bar{p}_1$ respectively.*

Proof. By induction. In the base case when $n = 1$, $\overline{\mathbf{Min}}(p_1) = p_1, \bar{p}_1$, so the claim holds. Inductive Hypothesis: For $n \geq 1$, assume that the first and last signed permutation in $\overline{\mathbf{Min}}(p_1p_2p_3 \cdots p_n)$ are $p_1p_2p_3 \cdots p_n$ and $\bar{p}_n \cdots \bar{p}_3\bar{p}_2\bar{p}_1$ respectively The first signed permutation of $\overline{\mathbf{Min}}(p_1p_2p_3 \cdots p_{n+1})$ is the first signed permutation of $\overline{\mathbf{Min}}(p_1p_2p_3 \cdots p_n) \cdot p_{n+1}$. Applying the inductive hypothesis this signed permutation is $p_1p_2p_3 \cdots p_{n+1}$. Similarly, the last signed permutation of $\overline{\mathbf{Min}}(p_1p_2p_3 \cdots p_{n+1})$ is the last signed permutation of $\overline{\mathbf{Min}}(p_2p_3 \cdots p_{n+1}) \cdot \bar{p}_1$. Applying the inductive hypothesis this signed permutation is $\bar{p}_{n+1} \cdots \bar{p}_3\bar{p}_2\bar{p}_1$. □

Lemma 5.3. *For $n \geq 1$, the flip-sequence for $\overline{\mathbf{Min}}(\mathbf{p})$ is $\overline{\sigma}_n$.*

Proof. By induction. In the base case $\overline{\mathbf{Min}}(p_1) = p_1, \bar{p}_1$ and the flip-sequence is $\overline{\sigma}_1 = 1$. Inductive Hypothesis: For $n \geq 1$ assume that the sequence of flips used to create $\overline{\mathbf{Min}}(p_1p_2p_3 \cdots p_n)$ is given by $\overline{\sigma}_n$. Consider $\overline{\mathbf{Min}}(\mathbf{p})$ where $\mathbf{p} = p_1p_2p_3 \cdots p_{n+1}$. By applying Lemma 5.2 we prove that the signed permutations between the $2(n+1)$ recursive listings of $\overline{\mathbf{Min}}(\mathbf{p})$ differ by a flip of length $n+1$. For $1 \leq i \leq n+1$, the last element of $\overline{\mathbf{Min}}(\mathbf{p}'(i)) \cdot p_i$ is $\bar{p}_{i-1} \cdots \bar{p}_1 p_{n+1} \cdots p_i$ and the first element of $\overline{\mathbf{Min}}(\mathbf{p}'(i-1)) \cdot p_{i-1}$ is $\bar{p}_i \cdots \bar{p}_{n+1} p_1 \cdots p_{i-1}$. These two signed permutations differ by a flip of length $n+1$. A similar result holds replacing $\mathbf{p}'(i)$ and $\mathbf{p}'(i-1)$ with $\bar{\mathbf{p}}(i)$ and $\bar{\mathbf{p}}(i-1)$. In the remaining case, the last element of $\overline{\mathbf{Min}}(\mathbf{p}'(1)) \cdot p_1$ is $p_{n+1} \cdots p_2 p_1$ which differs by a flip of length $n+1$ from the first element of $\overline{\mathbf{Min}}(\bar{\mathbf{p}}(n+1)) \cdot p_{n+1}$ which is $\bar{p}_1 \cdots \bar{p}_{n+1}$. Thus, by applying the inductive hypothesis, the flip-sequence for $\overline{\mathbf{Min}}(\mathbf{p})$ is $(\overline{\sigma}_n, n+1)^{2(n+1)-1}, \overline{\sigma}_n$ which is exactly $\overline{\sigma}_{n+1}$. □

Theorem 5.4. *For $n \geq 1$, the listing $\overline{\mathbf{Min}}(\mathbf{p})$ is a flip Gray code for signed permutations, where the first and last signed permutations differ by a flip of length n.*

Proof. From Lemma 5.3, the flip-sequence for $\overline{\mathbf{Min}}(\mathbf{p})$ is given by $\overline{\sigma}_n$. It is easy to see that the length of the flip-sequence $\overline{\sigma}_n$ is $2^n n! - 1$. Inductively, it is trivial to observe that each signed permutation of $\overline{\mathbf{Min}}(\mathbf{p})$ is unique. Thus, each of the $2^n n!$ signed permutations must be listed exactly once, making $\overline{\mathbf{Min}}(\mathbf{p})$ a signed permutation flip Gray code. Finally, from Lemma 5.2, the first and last signed permutations of the listing differ by a flip of length n. □

Lemma 5.5. *For $n \geq 1$, the flip-sequence for $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ is $\overline{\sigma}_n$.*

Proof. By contradiction. Suppose the sequence of flips used by $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ differs from $\overline{\sigma}_n$. Let j be the smallest value such that the j-th flip used to create $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ differs from the j-th value of $\overline{\sigma}_n$. Let these flip lengths be s and t respectively. Since $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ follows a greedy minimum-flip strategy and because $\overline{\sigma}_n$ produces a valid signed permutation Gray code by Theorem 5.4 where no signed permutation is repeated, it must be that $s < t$. Let $\mathbf{q} = q_1q_2q_3 \cdots q_n$ denote the j-th signed permutation in the listing $\overline{\mathsf{MinGreedy}}(\mathbf{p})$; it is the signed permutation prior to the j-th flip. By the recursive definition of $\overline{\sigma}_n$, the $2^{t-1}(t-1)!-1$ elements immediately prior to the j-th element in $\overline{\sigma}_n$ each have value less than t. This means that $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ would apply $2^{t-1}(t-1)!$ consecutive flips of length less than t. But this contradicts that fact that $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ produces a list of unique strings. □

Corollary 5.6. *For $n \geq 1$, the listings $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ and $\overline{\mathbf{Min}}(\mathbf{p})$ are equivalent.*

Proof. By definition $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ starts with signed permutation $\mathbf{p}$ and by Lemma 5.2 $\overline{\mathbf{Min}}(\mathbf{p})$ also starts with $\mathbf{p}$. Since they are each created by the same flip-sequence by Lemma 5.3 and Lemma 5.5, they will produce the same listing of signed permutations. □

To determine the average flip length in the listing $\overline{\mathsf{MinGreedy}}(\mathbf{p})$, let $\overline{\sigma}'_n$ denote the sequence $\overline{\sigma}_n$ with the added value n to account for the last flip required to return to the initial string. Observe that $\overline{\sigma}'_{n+1}$ corresponds to $2n$ copies of $\overline{\sigma}'_n$ with every $(2^n \cdot n!)$-th term incremented by 1. Thus, letting $avg(n)$ denote the average flip length in the sequence $\overline{\sigma}'_n$, we note that $avg(n+1) = avg(n) + \frac{1}{2^n n!}$. Taking the base case of $avg(1) = 1$ into account we obtain the following expression:

$$avg(n) = \sum_{j=0}^{n-1} \frac{1}{2^j j!}.$$

Taking the final sum to infinity yields the well-known Maclaurin series expansion of e^x when $x = 1/2$. Thus, as n goes to infinity the average flip length approaches $\sqrt{e} \approx 1.64872127$.

6. Maximum Flips for Permutations

In this section we study the maximum flip greedy algorithm and prove that it exhaustively lists all unsigned permutations by deriving an equivalent recursive formulation. We begin by looking at the greedy listings $\mathsf{MaxGreedy}(1234)$ and $\mathsf{MaxGreedy}(12345)$. The length of the flip to go from one permutation to the next is given in parentheses after each permutation.

Example 6.1. $\mathsf{MaxGreedy}(1234)$ *(read down, then left to right):*

1234 (4)	2314 (4)	3124 (4)
4321 (3)	4132 (3)	4213 (3)
2341 (4)	3142 (4)	1243 (4)
1432 (3)	2413 (3)	3421 (3)
3412 (4)	1423 (4)	2431 (4)
2143 (3)	3241 (3)	1342 (3)
4123 (4)	4231 (4)	4312 (4)
3214 (2)	1324 (2)	2134 (2)

Example 6.2. $\mathsf{MaxGreedy}(12345)$ *(read down, then left to right):*

12345 (5)	23415 (5)	34125 (5)	41235 (5)	23145 (5)	31425 (5)	14235 (5)	42315 (5)	31245 (5)	12435 (5)	24315 (5)	43125 (5)
54321 (4)	51432 (4)	52143 (4)	53214 (4)	54132 (4)	52413 (4)	53241 (4)	51324 (4)	54213 (4)	53421 (4)	51342 (4)	52134 (4)
23451 (5)	34152 (5)	41253 (5)	12354 (5)	31452 (5)	14253 (5)	42351 (5)	23154 (5)	12453 (5)	24351 (5)	43152 (5)	31254 (5)
15432 (4)	25143 (4)	35214 (4)	45321 (4)	25413 (4)	35241 (4)	15324 (4)	45132 (4)	35421 (4)	15342 (4)	25134 (4)	45213 (4)
34512 (5)	41523 (5)	12534 (5)	23541 (5)	14523 (5)	42531 (5)	23514 (5)	31542 (5)	24531 (5)	43512 (5)	31524 (5)	12543 (5)
21543 (4)	32514 (4)	43521 (4)	14532 (4)	32541 (4)	13524 (4)	41532 (4)	24513 (4)	13542 (4)	21534 (4)	42513 (4)	34521 (4)
45123 (5)	15234 (5)	25341 (5)	35412 (5)	45231 (5)	25314 (5)	35142 (5)	15423 (5)	45312 (5)	35124 (5)	15243 (5)	25431 (5)
32154 (4)	43251 (4)	14352 (4)	21453 (4)	13254 (4)	41352 (4)	24153 (4)	32451 (4)	21354 (4)	42153 (4)	34251 (4)	13452 (4)
51234 (5)	52341 (5)	53412 (5)	54123 (5)	52314 (5)	53142 (5)	51423 (5)	54231 (5)	53124 (5)	51243 (5)	52431 (5)	54312 (5)
43215 (3)	14325 (3)	21435 (3)	32145 (2)	41325 (3)	24135 (3)	32415 (3)	13245 (2)	42135 (3)	34215 (3)	13425 (3)	21345 (2)

In each example, observe that every second flip has length n. Thus, when deriving a recurrence for the sequence of flips required to generate the greedy maximum-flip listing, we start by deriving a recurrence for the length of every second flip. To derive this recurrence, the important flip to consider is the one happening at the bottom of each column in the examples. Observe that the flip lengths at the bottom of each column in the example for $\mathsf{MaxGreedy}(12345)$ correspond to every second flip length in the example for $\mathsf{MaxGreedy}(1234)$. Letting $\tau'_n = t_1, t_2, \ldots, t_j$ consider the following recurrence:

$$\tau'_{n+1} = \begin{cases} 2, 2 & \text{if } n+1 = 3 \\ n^n, t_1, n^n, t_2, \ldots, n^n, t_j, n^n & \text{if } n+1 > 3. \end{cases}$$

Lemma 6.3. *For $n \geq 3$, the number of elements in the sequence τ'_n is $\frac{n!}{2} - 1$.*

Proof. By induction. In the base case when $n = 3$, τ'_n has 2 elements and $\frac{3!}{2} - 1 = 2$. Inductively, it is easy to see that the number of elements in τ'_{n+1} is $(n+1) \cdot (\frac{(n)!}{2} - 1) + n = \frac{(n+1)!}{2} - 1$. □

Letting $\tau'_n = t_1, t_2, \ldots, t_j$, we will show that the sequence of flips used to create $\mathsf{MaxGreedy}(\mathbf{p})$ is given by σ'_n which is defined as follows:

$$\sigma'_n = \begin{cases} 2 & \text{if } n = 2 \\ n, t_1, n, t_2, \ldots, n, t_j, n & \text{if } n > 2. \end{cases}$$

Before we can prove this claim, we need to better understand the permutation ordering. Again, by considering the examples for MaxGreedy(1234) and MaxGreedy(12345), observe that the permutations in each column are closed under rotation and reversal: they form a *bracelet class*. The rotation effect is easily observed since:

- ▷ applying flip_n followed by flip_{n-1} rotates a permutation one position to the left and
- ▷ applying flip_{n-1} followed by flip_n rotates a permutation one position to the right.

These bracelet sequences form the crux of a recursive formulation for MaxGreedy($\mathbf{p}$). Define the *bracelet sequence* of permutation $\mathbf{p_1} = p_1p_2p_3 \cdots p_n$, where $n \geq 3$ as:

$$\mathsf{brace}(\mathbf{p_1}) = \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \ldots, \mathbf{p_{2n}} \text{ such that } \mathbf{p_i} = \begin{cases} \mathsf{flip}_n(\mathbf{p_{i-1}}) & \text{if } i \text{ is even} \\ \mathsf{flip}_{n-1}(\mathbf{p_{i-1}}) & \text{if } i > 1 \text{ is odd.} \end{cases}$$

The permutation $\mathbf{p_1}$ is called the *representative* of the bracelet sequence brace($\mathbf{p_1}$). Since the order alternates flips of lengths n and $n-1$, the odd permutations $\mathbf{p_1}, \mathbf{p_3}, \mathbf{p_5}, \ldots, \mathbf{p_{2n-1}}$ are all rotations of $\mathbf{p_1}$ and the even permutations $\mathbf{p_2}, \mathbf{p_4}, \mathbf{p_6}, \ldots, \mathbf{p_{2n}}$ are all rotations of the reversal of $\mathbf{p_1}$ which is $\mathsf{flip}_n(\mathbf{p_1}) = \mathbf{p_2}$. Thus, the permutations in each bracelet sequence form a bracelet class.

Remark 6.4. *The last permutation in* brace($\mathbf{p}$) *is* $\mathsf{flip}_{n-1}(\mathbf{p})$.

Remark 6.5. *There are exactly two permutations in* brace($p_1p_2p_3 \cdots p_n$) *that end with* p_n*, namely* $p_1p_2p_3 \cdots p_n$ *and* $p_{n-1} \cdots p_3p_2p_1p_n$*, and they differ by* flip_{n-1}.

Now focus on the order of the bracelets from the examples. Notice that every second permutation in MaxGreedy(1234), starting with the first permutation, corresponds to the first 4 characters of the first permutation in each column in the example for MaxGreedy(12345) – <u>1234</u>5, <u>2341</u>5, <u>3412</u>5, ..., <u>4312</u>5. This illustrates how the listing MaxGreedy($\mathbf{p}$), given permutation $\mathbf{p} = p_1p_2p_3 \cdots p_n$, can be understood recursively. Consider the following sequence $\mathbf{Max}(\mathbf{p})$, defined by the recurrence:

$$\mathbf{Max}(\mathbf{p}) = \begin{cases} p_1 & \text{if } n = 1 \\ p_1p_2, p_2p_1 & \text{if } n = 2 \\ \mathsf{brace}(\mathbf{q_1} \cdot p_n), \mathsf{brace}(\mathbf{q_3} \cdot p_n), \mathsf{brace}(\mathbf{q_5} \cdot p_n), \ldots, \mathsf{brace}(\mathbf{q_{m-1}} \cdot p_n) & \text{if } n \geq 3, \end{cases} \quad (3)$$

where $\mathbf{Max}(p_1p_2p_3 \cdots p_{n-1}) = \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \ldots, \mathbf{q_m}$. The following lemma shows that $\mathbf{m} = (n-1)!$.

Lemma 6.6. *The number of elements in the sequence* $\mathbf{Max}(p_1p_2p_3 \cdots p_n)$ *is* $n!$.

Proof. By induction. The base cases are clearly satisfied for $n = 1$ and $n = 2$. Inductive Hypothesis: For $n \geq 2$, assume the claim is true. Consider $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$. By the inductive hypothesis $\mathbf{Max}(p_1p_2p_3 \cdots p_n)$ has $n!$ elements which is an even number since $n \geq 2$. Thus, from the recursive definition, $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ is the concatenation of $n!/2$ bracelet sequences of length $2(n+1)$. Thus the total number of permutations in the sequence $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ is $(n+1)!$. □

Our goal is to prove that MaxGreedy($\mathbf{p}$) and $\mathbf{Max}(\mathbf{p})$ are equivalent flip Gray code listings for permutations.

Lemma 6.7. *For $n \geq 2$, the first, last, and second last permutations in the listing* $\mathbf{Max}(\mathbf{p})$ *are* $\mathbf{p}$*,* $\mathsf{flip}_2(\mathbf{p})$*, and* $\mathsf{flip}_n(\mathsf{flip}_2(\mathbf{p}))$ *respectively.*

Proof. By induction. In the base case when $n = 2$, $\mathbf{Max}(p_1p_2) = p_1p_2, p_2p_1$ satisfying the claim. Inductive Hypothesis: For $n \geq 2$, assume the claim is true. From its recurrence, the first permutation of $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ is the first permutation of the bracelet sequence $\mathsf{brace}(\mathbf{q_1} \cdot p_{n+1})$, which by definition is $\mathbf{q_1} \cdot p_{n+1}$, where $\mathbf{q_1}$ is the first permutation in $\mathbf{Max}(p_1p_2p_3 \cdots p_n)$. Applying the inductive hypothesis, $\mathbf{q_1} \cdot p_{n+1}$ is $p_1p_2p_3 \cdots p_{n+1}$.

The last permutation of $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ is the last permutation of the bracelet sequence $\mathsf{brace}(\mathbf{q_{m-1}} \cdot p_{n+1})$, where $\mathbf{q_{m-1}}$ is the *second last* permutation in $\mathbf{Max}(p_1p_2p_3 \cdots p_n)$ – by its recursive definition note that $\mathbf{m}$ is in fact even. By the inductive hypothesis, $\mathbf{q_{m-1}}$ is $p_np_{n-1} \cdots p_3p_1p_2$. Thus the last permutation of $\mathsf{brace}(\mathbf{q_{m-1}} \cdot p_{n+1})$ is $p_2p_1p_3p_4 \cdots p_{n+1}$, since by Remark 6.4 the first and last permutations of the bracelet sequence differ by a flip of length n. Finally, each bracelet sequence in the recursive definition of $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ contains at least two permutations since $n \geq 2$. Thus, the second last permutation of $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ differs by a flip of length $n+1$ from its last permutation. ☐

Lemma 6.8. *For $n \geq 2$, the flip-sequence for* $\mathbf{Max}(\mathbf{p})$ *is σ'_n.*

Proof. By induction. In the base case $\mathbf{Max}(p_1p_2) = p_1p_2, p_2p_1$ and the flip-sequence is $\sigma'_2 = 2$. Inductive Hypothesis: For $n \geq 2$ assume that the sequence of flips used to create $\mathbf{Max}(p_1p_2 \cdots p_n)$ is given by σ'_n. Let $\mathbf{Max}(p_1p_2p_3 \cdots p_n) = \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \ldots, \mathbf{q_m}$ and let $\tau'_n = t_1, t_2, \ldots, t_j$. From the inductive hypothesis $\mathbf{q_{2i+1}} = \mathsf{flip}_{t_i}(\mathsf{flip}_n(\mathbf{q_{2i-1}}))$ for $1 \leq i \leq j$.

Now consider the recursive definition of $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$. By definition of a bracelet sequence, the first permutation in $\mathsf{brace}(\mathbf{q} \cdot p_{n+1})$ is $\mathbf{q} \cdot p_{n+1}$ and the last permutation by Remark 6.4 is $\mathsf{flip}_n(\mathbf{q} \cdot p_{n+1})$ for any $\mathbf{q} \in \{\mathbf{q_1}, \mathbf{q_3}, \ldots, \mathbf{q_{m-1}}\}$. Thus, the last permutation in $\mathsf{brace}(\mathbf{q_{2i-1}} \cdot p_{n+1})$ differs from the first permutation in $\mathsf{brace}(\mathbf{q_{2i+1}} \cdot p_{n+1})$ by a flip of length t_i.

By the definition of a bracelet sequence, every second flip in $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$, starting from the first permutation, has length $n+1$. Every second flip starting from the second permutation is given by the sequence

$$n^n, t_1, n^n, t_2, \ldots, n^n, t_j, n^n$$

which is τ'_{n+1}. Thus the sequence of flips used to generate the listing $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ is σ'_{n+1}. ☐

Lemma 6.9. *For $n \geq 2$, the listings* $\mathsf{MaxGreedy}(\mathbf{p})$ *and* $\mathbf{Max}(\mathbf{p})$ *are equivalent.*

Proof. By definition, $\mathsf{MaxGreedy}(\mathbf{p})$ starts with permutation $\mathbf{p}$ and by Lemma 6.7 $\mathbf{Max}(\mathbf{p})$ also starts with $\mathbf{p}$. Thus, we must prove that the flip-sequence used by $\mathbf{Max}(\mathbf{p})$ is greedy maximum. This is done by induction on n.

In the base case, $\mathbf{Max}(p_1p_2) = p_1p_2, p_2p_1$ is greedy maximum. Inductive Hypothesis: the sequence of flips used by $\mathbf{Max}(p_1p_2p_3 \cdots p_n)$ is greedy maximum for $n \geq 2$. Let $\mathbf{Max}(p_1p_2p_3 \cdots p_n) = \mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_m}$. Since σ'_n corresponds to its flip-sequence by Lemma 6.8, every second flip used to generate this sequence has length n, which implies that $\mathbf{q_{2i}} = \mathsf{flip}_n(\mathbf{q_{2i-1}})$ for $1 \leq i \leq m/2$.

Now consider $\mathbf{Max}(p_1p_2p_3 \cdots p_{n+1})$ and its recursive definition. Note that successive flips in any bracelet sequence $\mathsf{brace}(\mathbf{q})$ are clearly greedy maximum. Thus, it suffices to show for $1 \leq i < m/2$ that the last permutation $\mathbf{x}$ in $\mathsf{brace}(\mathbf{q_{2i-1}} \cdot p_{n+1})$, uses a greedy maximum flip to obtain the first permutation $\mathbf{y}$ in $\mathsf{brace}(\mathbf{q_{2i+1}} \cdot p_{n+1})$.

- From Remark 6.4, $\mathbf{x} = \mathsf{flip}_n(\mathbf{q_{2i-1}} \cdot p_{n+1}) = \mathsf{flip}_n(\mathbf{q_{2i-1}}) \cdot p_{n+1} = \mathbf{q_{2i}} \cdot p_{n+1}$.
- By the definition of a bracelet sequence, $\mathbf{y} = \mathbf{q_{2i+1}} \cdot p_{n+1}$.
- By the inductive hypothesis, $\mathbf{q_{2i}}$ differs from $\mathbf{q_{2i+1}}$ by a greedy maximum flip of length l. This implies that $\mathsf{flip}_l(\mathbf{x}) = \mathbf{y}$.

We must show that l is the greedy maximum flip length from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbf{Max}(p_1p_2p_3\cdots p_{n+1})$. Since $\mathbf{x}$ is the last permutation in a bracelet sequence of $\mathbf{Max}(p_1p_2p_3\cdots p_{n+1})$, it is at an even index in the listing. Thus, $\mathsf{flip}_{n+1}(\mathbf{x})$, yields the previous permutation by the definition of a bracelet sequence and hence the greedy maximum flip to go from $\mathbf{x}$ to $\mathbf{y}$ must be less than $n+1$.

Now, consider $\mathsf{flip}_k(\mathbf{x})$ for some $l < k < n+1$. From the greedy maximum choice of l, observe that $\mathsf{flip}_k(\mathbf{q_{2i}})$ comes before $\mathbf{q_{2i+1}}$ in $\mathbf{Max}(p_1p_2p_3\cdots p_n)$, say at position $\mathbf{t}$. If $\mathbf{t}$ is odd, then by the recursive definition of $\mathbf{Max}(p_1p_2p_3\cdots p_{n+1})$, $\mathsf{flip}_k(\mathbf{x}) = \mathbf{q_t}\cdot_{n+1}$ appears before $\mathbf{y}$ in the listing as the first permutation of some bracelet sequence. If $\mathbf{t}$ is even, then $\mathbf{t}-1$ is odd and hence $\mathbf{q_{t-1}} \cdot p_{n+1}$ appears before $\mathbf{y}$ as a bracelet sequence representative. As noted earlier, the last permutation in such a bracelet sequence is $\mathbf{q_t} \cdot p_{n+1}$, and hence it appears before $\mathbf{y}$. Thus l is a greedy maximum-flip to go from $\mathbf{x}$ to $\mathbf{y}$ in this listing. ☐

Theorem 6.10. *For $n \geq 2$, the listing $\mathbf{Max}(\mathbf{p})$ is a flip Gray code for permutations, where the first and last permutations differ by a flip of length 2.*

Proof. By applying Lemma 6.3, the length of the flip-sequence σ'_n is $n!-1$. Thus, since the flip-sequence for $\mathbf{Max}(\mathbf{p})$ is σ'_n by Lemma 6.8, the number of permutations in the listing $\mathbf{Max}(\mathbf{p})$ is $n!$. Since the listing $\mathbf{Max}(\mathbf{p})$ is equivalent to the greedy listing $\mathsf{MaxGreedy}(\mathbf{p})$ by Lemma 6.9, each permutation in the listing must be unique. Thus, $\mathbf{Max}(\mathbf{p})$ is a permutation flip Gray code. Finally, from Lemma 6.7, the first and last permutation of the listing differ by a flip of length 2. ☐

Next we give an analysis on the average flip length, denoted $avg(n)$ required to generate $\mathbf{Max}(\mathbf{p})$. We consider the ordering to be circular to slightly simplify the analysis, and hence the average includes the final flip of length 2 to go from the last signed permutation to the first one. An upper bound on this average is obtained by bounding each element in the sequence that is less than or equal to $n-1$ by $n-1$. Since there are $\frac{n!}{2}$ occurrences of n in σ'_n, we obtain the following upper bound:

$$avg(n) \;\leq\; \frac{1}{n!}\left(n \cdot \frac{n!}{2} \;+\; (n-1)\cdot\frac{n!}{2}\right) \;=\; n - \frac{1}{2}.$$

To obtain a lower bound for $n \geq 5$, we ignore all flips of length less than $n-2$. Observe that there are $(\frac{n-1}{n})\frac{n!}{2}$ occurrences of $n-1$ and $(\frac{n-2}{n-1})\frac{(n-1)!}{2}$ occurrences of $n-2$ in τ'_n (and hence σ'_n). Thus:

$$\begin{aligned}
avg(n) &\geq \frac{1}{n!}\left(n\cdot\frac{n!}{2} + (n-1)\cdot\frac{n-1}{n}\cdot\frac{n!}{2} + (n-2)\cdot\frac{n-2}{n-1}\cdot\frac{(n-1)!}{2}\right) \\
&= \frac{n}{2} + \frac{(n-1)^2}{2n} + \frac{(n-2)^2}{2n\cdot(n-1)} \\
&> \frac{n}{2} + \frac{(n^2-2n)}{2n} + \frac{(n^2-4n)}{2n^2} \\
&= \frac{n}{2} + \frac{n}{2} - 1 + \frac{1}{2} - \frac{2}{n} \\
&= n - \frac{1}{2} - \frac{2}{n}.
\end{aligned}$$

As n goes to infinity the average flip length approaches $n - \frac{1}{2}$.

7. Maximum Flips for Signed Permutations

In this section we study the maximum flip greedy algorithm and prove that it exhaustively lists all signed permutations by deriving an equivalent recursive formulation. We begin by looking at the greedy listing $\overline{\mathsf{MaxGreedy}}(123)$. The length of the flip to go from one signed permutation to the next is given in parentheses after each signed permutation.

Example 7.1. $\overline{\mathsf{MaxGreedy}}(123)$ *(read down, then left to right):*

123 (3)	$2\bar{1}3$ (3)	$\bar{1}\bar{2}3$ (3)	$\bar{2}13$ (3)
$\bar{3}\bar{2}\bar{1}$ (2)	$\bar{3}1\bar{2}$ (2)	$\bar{3}21$ (2)	$\bar{3}\bar{1}2$ (2)
$23\bar{1}$ (3)	$\bar{1}3\bar{2}$ (3)	$\bar{2}31$ (3)	132 (3)
$1\bar{3}\bar{2}$ (2)	$2\bar{3}1$ (2)	$\bar{1}\bar{3}2$ (2)	$\bar{2}\bar{3}\bar{1}$ (2)
$3\bar{1}\bar{2}$ (3)	$3\bar{2}1$ (3)	312 (3)	$32\bar{1}$ (3)
$21\bar{3}$ (2)	$\bar{1}2\bar{3}$ (2)	$\bar{2}\bar{1}\bar{3}$ (2)	$1\bar{2}\bar{3}$ (2)
$\bar{1}\bar{2}\bar{3}$ (3)	$\bar{2}1\bar{3}$ (3)	$12\bar{3}$ (3)	$2\bar{1}\bar{3}$ (3)
321 (2)	$3\bar{1}2$ (2)	$3\bar{2}\bar{1}$ (2)	$31\bar{2}$ (2)
$\bar{2}\bar{3}1$ (3)	$1\bar{3}2$ (3)	$2\bar{3}\bar{1}$ (3)	$\bar{1}\bar{3}\bar{2}$ (3)
$\bar{1}32$ (2)	$\bar{2}3\bar{1}$ (2)	$13\bar{2}$ (2)	231 (2)
$\bar{3}12$ (3)	$\bar{3}2\bar{1}$ (3)	$\bar{3}\bar{1}\bar{2}$ (3)	$\bar{3}\bar{2}1$ (3)
$\bar{2}\bar{1}3$ (1)	$1\bar{2}3$ (1)	213 (1)	$\bar{1}23$ (1)

As with the maximum flip greedy algorithm for unsigned permutations, observe that every second flip (starting from the first signed permutation) has length n. Thus, we start by deriving a recurrence for the length of every second flip, starting from the second signed permutation in the ordering. To derive this recurrence, the important flip to consider is the one happening at the bottom of each column in the examples. The flip lengths at the bottom of each column in the example for $\overline{\mathsf{MaxGreedy}}(123)$ correspond to every second flip in the sequence for $\overline{\mathsf{MaxGreedy}}(12)$ which is given by 2,**1**,2,**1**,2,**1**,2,**1** when considered circularly.

Let $\overline{\tau'}_n = t_1, t_2, \ldots, t_j$ and consider the following recurrence:

$$\overline{\tau'}_{n+1} = \begin{cases} 1, 1, 1 & \text{if } n+1 = 2 \\ n^{2n+1}, t_1, n^{2n+1}, t_2, \ldots, n^{2n+1}, t_j, n^{2n+1} & \text{if } n+1 > 2. \end{cases}$$

Lemma 7.2. *For $n \geq 2$, the number of elements in the sequence $\overline{\tau'}_n$ is $2^{n-1}n! - 1$.*

Proof. By induction. In the base case when $n = 2$, $\overline{\tau'}_n$ has 3 elements and $2^1 2! - 1 = 3$. Inductively, it is easy to see that the number of elements in $\overline{\tau'}_{n+1}$ is $(2n+2) \cdot (2^{n-1}n! - 1) + 2n + 1 = 2^n(n+1)! - 1$. ☐

Letingt $\overline{\tau'}_n = t_1, t_2, \ldots, t_j$, we will show that the sequence of flips used to create $\overline{\mathsf{MaxGreedy}}(\mathbf{p})$ is given by $\overline{\sigma'}_n$ which is defined as follows:

$$\overline{\sigma'}_n = \begin{cases} 1 & \text{if } n = 1 \\ n, t_1, n, t_2, \ldots, n, t_j, n & \text{if } n > 1. \end{cases}$$

Before we can prove this claim, we need to better understand the signed permutation ordering. Observe that each column in the example for $\overline{\mathsf{MaxGreedy}}(123)$ has *signed* groupings similar to the unsigned case, but of size $4n$ compared to $2n$ in the unsigned case. Define the *signed bracelet sequence* of a signed permutation $\mathbf{p_1} = p_1p_2p_3\cdots p_n$, where $n \geq 2$ as:

$$\overline{\mathsf{brace}}(\mathbf{p_1}) = \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \ldots, \mathbf{p_{4n}} \text{ such that } \mathbf{p_i} = \begin{cases} \overline{\mathsf{flip}}_n(\mathbf{p_{i-1}}) & \text{if } i \text{ is even} \\ \overline{\mathsf{flip}}_{n-1}(\mathbf{p_{i-1}}) & \text{if } i > 1 \text{ is odd.} \end{cases}$$

As with the unsigned case, the signed permutation $\mathbf{p_1}$ is called the *representative* of the signed bracelet sequence $\mathsf{brace}(\mathbf{p_1})$. Observe that applying a flip of size n followed by a flip of size $n-1$ to any signed permutation rotates the values one position to the left, changing the sign of the element that moved to the end. Repeating such a rotation n times, we obtain the original signed permutation with all the signs flipped. Repeating such a rotation $2n$ times returns us to the original starting signed permutation. From these observations we make the following remarks.

Remark 7.3. *The last signed permutation in a signed bracelet sequence* $\overline{\mathsf{brace}}(\mathbf{p_1})$ *is* $\overline{\mathsf{flip}}_{n-1}(\mathbf{p_1})$.

Lemma 7.4. *If* $\overline{\mathsf{brace}}(\mathbf{p_1}) = \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \ldots, \mathbf{p_{4n}}$ *where* $n \geq 2$ *then* $\mathsf{flipSign}(\mathbf{p_i}) = \mathbf{p_{2n+i}}$ *for* $1 \leq i \leq 2n$.

Proof. Consider a signed permutation $\mathbf{p} = p_1p_2\cdots p_n$. The result of applying a flip of size n followed by a flip of size $n-1$ is $p_2p_3\cdots p_n\bar{p}_1$ which is a rotation of $\mathbf{p}$ to the left and flipping the sign of the new last element. By repeatedly applying these two successive operations n times, the resulting permutation is $\bar{p}_1\bar{p}_2\cdots\bar{p}_n$. Thus, by the definition of a signed bracelet sequence, when i is odd, $\mathsf{flipSign}(\mathbf{p_i}) = \mathbf{p_{2n+i}}$ for $1 \leq i \leq 2n$.

The result of applying a flip of size $n-1$ followed by a flip of size n is $\bar{p}_np_1p_2\cdots p_{n-1}$ which is a rotation of $\mathbf{p}$ to the right and flipping the sign of the new first element. By repeatedly applying these two successive operations n times, the resulting permutation is $\bar{p}_1\bar{p}_2\cdots\bar{p}_n$. Thus, by the definition of a signed bracelet sequence, when i is even, $\mathsf{flipSign}(\mathbf{p_i}) = \mathbf{p_{2n+i}}$ for $1 \leq i \leq 2n$. ☐

Remark 7.5. *There are exactly two signed permutations in* $\overline{\mathsf{brace}}(p_1p_2p_3\cdots p_n)$ *that end with* p_n*, namely* $p_1p_2p_3\cdots p_n$ *and* $\bar{p}_{n-1}\cdots\bar{p}_3\bar{p}_2\bar{p}_1p_n$*, and they differ by* flip_{n-1}.

Using the definition for signed bracelet sequences, we arrive at a recurrence for the sequence $\overline{\mathbf{Max}}(\mathbf{p})$ similar to the one for the unsigned case. If $\mathbf{p} = p_1p_2p_3\cdots p_n$ is a signed permutation, then:

$$\overline{\mathbf{Max}}(\mathbf{p}) = \begin{cases} p_1, \bar{p}_1 & \text{if } n = 1 \\ \overline{\mathsf{brace}}(\mathbf{q_1}\cdot p_n), \overline{\mathsf{brace}}(\mathbf{q_3}\cdot p_n), \overline{\mathsf{brace}}(\mathbf{q_5}\cdot p_n), \ldots, \overline{\mathsf{brace}}(\mathbf{q_{m-1}}\cdot p_n) & \text{if } n \geq 2, \end{cases} \tag{4}$$

where $\overline{\mathbf{Max}}(p_1p_2p_3\cdots p_{n-1}) = \mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_m}$. The following lemma shows that $\mathbf{m} = 2^{n-1}(n-1)!$.

Lemma 7.6. *The number of elements in the sequence* $\overline{\mathbf{Max}}(p_1p_2p_3\cdots p_n)$ *is* $2^nn!$.

Proof. By induction. The base case is clearly satisfied for $n = 1$. Inductive Hypothesis: For $n \geq 1$, assume the claim is true. Consider $\overline{\mathbf{Max}}(p_1p_2p_3\cdots p_{n+1})$. By the inductive hypothesis $\overline{\mathbf{Max}}(p_1p_2p_3\cdots p_n)$ has $2^nn!$ elements which is an even number since $n \geq 1$. Thus, from the recursive definition, $\overline{\mathbf{Max}}(p_1p_2p_3\cdots p_{n+1})$ is the concatenation of $2^{n-1}n!$ signed bracelet sequences of length $4(n+1)$. Thus, the total number of signed permutations in the sequence $\overline{\mathbf{Max}}(p_1p_2p_3\cdots p_{n+1})$ is $2^{n+1}(n+1)!$. ☐

Our goal is to show that $\overline{\mathsf{MaxGreedy}}(\mathbf{p})$ and $\overline{\mathbf{Max}}(\mathbf{p})$ are equivalent flip Gray code listings for signed permutations.

Lemma 7.7. *For $n \geq 1$, the first, last, and second last signed permutations in the listing $\overline{\mathbf{Max}}(\mathbf{p})$ are $\mathbf{p}$, $\overline{\mathsf{flip}}_1(\mathbf{p})$, and $\overline{\mathsf{flip}}_n(\overline{\mathsf{flip}}_1(\mathbf{p}))$ respectively.*

Proof. By induction. In the base case when $n = 1$, $\overline{\mathbf{Max}}(p_1) = p_1, \bar{p}_1$ satisfying the claim. Inductive Hypothesis: For $n \geq 1$, assume the claim is true. From its recurrence, the first signed permutation of $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$ is the first signed permutation of the signed bracelet sequence $\overline{\mathsf{brace}}(\mathbf{q_1} \cdot p_{n+1})$, which by definition is $\mathbf{q_1} \cdot p_{n+1}$, where $\mathbf{q_1}$ is the first signed permutation in $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_n)$. Applying the inductive hypothesis, $\mathbf{q_1} \cdot p_{n+1}$ is $p_1p_2p_3 \cdots p_{n+1}$.

The last signed permutation of $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$ is the last signed permutation of the bracelet sequence $\overline{\mathsf{brace}}(\mathbf{q_{m-1}} \cdot p_{n+1})$, where $\mathbf{q_{m-1}}$ is the *second last* signed permutation in $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_n)$ (by its recursive definition note that $\mathbf{m}$ is in fact even). By the inductive hypothesis, $\mathbf{q_{m-1}}$ is $\bar{p}_n \cdots \bar{p}_3\bar{p}_2p_1$. Thus the last signed permutation of $\overline{\mathsf{brace}}(\mathbf{q_{m-1}} \cdot p_{n+1})$ is $\bar{p}_1p_2p_3p_4 \cdots p_{n+1}$, since by Remark 7.3 the first and last signed permutations of the signed bracelet sequence differ by a flip of length n. Finally, each signed bracelet sequence in the recursive definition of $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$ contains at least two signed permutations. Thus, the second last signed permutation of $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$ differs by a flip of length $n + 1$ from its last signed permutation. ☐

Lemma 7.8. *For $n \geq 1$, the flip-sequence for $\overline{\mathbf{Max}}(\mathbf{p})$ is $\overline{\sigma'}_n$.*

Proof. By induction. In the base case when $n = 1$, $\overline{\mathbf{Max}}(p_1) = p_1, \bar{p}_1$ and the flip-sequence is $\overline{\sigma'}_1 = 1$. Inductive Hypothesis: For $n \geq 1$ assume that the sequence of flips used to create $\overline{\mathbf{Max}}(p_1p_2 \cdots p_n)$ is given by $\overline{\sigma'}_n$. Let $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_n) = \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}, \ldots, \mathbf{q_m}$ and let $\overline{\tau'}_n = t_1, t_2, \ldots, t_j$. From the inductive hypothesis $\mathbf{q_{2i+1}} = \overline{\mathsf{flip}}_{t_i}(\overline{\mathsf{flip}}_n(\mathbf{q_{2i-1}}))$ for $1 \leq i \leq j$.

Now consider the recursive definition of $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$. By definition of a signed bracelet sequence, the first signed permutation in $\overline{\mathsf{brace}}(\mathbf{q} \cdot p_{n+1})$ is $\mathbf{q} \cdot p_{n+1}$ and the last signed permutation by Remark 7.3 is $\overline{\mathsf{flip}}_n(\mathbf{q} \cdot p_{n+1})$ for any $\mathbf{q} \in \{\mathbf{q_1}, \mathbf{q_3}, \ldots, \mathbf{q_{m-1}}\}$. Thus, the last signed permutation in $\overline{\mathsf{brace}}(\mathbf{q_{2i-1}} \cdot p_{n+1})$ differs from the first signed permutation in $\overline{\mathsf{brace}}(\mathbf{q_{2i+1}} \cdot p_{n+1})$ by a flip of length t_i.

By the definition of a bracelet sequence, every second flip in $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$, starting from the first signed permutation, has length $n+1$. Every second flip starting from the second signed permutation is given by the sequence

$$n^{2n+1}, t_1, n^{2n+1}, t_2, \ldots, n^{2n+1}, t_j$$

which is $\overline{\tau'}_{n+1}$. Thus the sequence of flips used to generate the listing $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$ is $\overline{\sigma'}_{n+1}$. ☐

Lemma 7.9. *For $n \geq 1$, the listings $\overline{\mathsf{MaxGreedy}}(\mathbf{p})$ and $\overline{\mathbf{Max}}(\mathbf{p})$ are equivalent.*

Proof. By definition, $\overline{\mathsf{MaxGreedy}}(\mathbf{p})$ starts with $\mathbf{p}$ and by Lemma 7.7 $\overline{\mathbf{Max}}(\mathbf{p})$ also starts with $\mathbf{p}$. Thus, we prove that the flip-sequence used by $\overline{\mathbf{Max}}(\mathbf{p})$ is greedy maximum. This is done by induction on n.

In the base case, $\overline{\mathbf{Max}}(p_1) = p_1, \bar{p}_1$ is greedy maximum. Inductive Hypothesis: For $n \geq 1$, the sequence of flips used by $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_n)$ is greedy maximum. Let $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_n) = \mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_m}$. Since $\overline{\sigma'}_n$ corresponds to its flip-sequence by Lemma 7.8, every second flip used to generate this sequence has length n, which implies that $\mathbf{q_{2i}} = \overline{\mathsf{flip}}_n(\mathbf{q_{2i-1}})$ for $1 \leq i \leq m/2$.

Now consider $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$ and its recursive definition. Successive flips in any signed bracelet sequence $\overline{\mathsf{brace}}(\mathbf{q})$ are clearly greedy maximum. Thus, it suffices to show for $1 \leq i < m/2$ that the last signed permutation $\mathbf{x}$ in $\overline{\mathsf{brace}}(\mathbf{q_{2i-1}} \cdot p_{n+1})$, uses a greedy maximum flip of obtain the first signed permutation $\mathbf{y}$ in $\overline{\mathsf{brace}}(\mathbf{q_{2i+1}} \cdot p_{n+1})$.

- From Remark 7.3, $\mathbf{x} = \overline{\mathsf{flip}}_n(\mathbf{q_{2i-1}} \cdot p_{n+1}) = \overline{\mathsf{flip}}_n(\mathbf{q_{2i-1}}) \cdot p_{n+1} = \mathbf{q_{2i}} \cdot p_{n+1}$.

- By the definition of a signed bracelet sequence, $\mathbf{y} = \mathbf{q_{2i+1}} \cdot p_{n+1}$.
- By the inductive hypothesis, $\mathbf{q_{2i}}$ differs from $\mathbf{q_{2i+1}}$ by a greedy maximum flip of length l. This implies that $\overline{\mathsf{flip}}_l(\mathbf{x}) = \mathbf{y}$.

We must show that l is the greedy maximum flip length from $\mathbf{x}$ to $\mathbf{y}$ in $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$. Since $\mathbf{x}$ is the last signed permutation in a signed bracelet sequence of $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$, it is at an even index in the listing. Thus, $\overline{\mathsf{flip}}_{n+1}(\mathbf{x})$, yields the previous signed permutation by the definition of a signed bracelet sequence and hence the greedy maximum flip to go from $\mathbf{x}$ to $\mathbf{y}$ must be less than $n+1$.

Now, consider $\overline{\mathsf{flip}}_k(\mathbf{x})$ for some $l < k < n+1$. From the greedy maximum choice of l, observe that $\overline{\mathsf{flip}}_k(\mathbf{q_{2i}})$ comes before $\mathbf{q_{2i+1}}$ in $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_n)$, say at position $\mathbf{t}$. If $\mathbf{t}$ is odd, then by the recursive definition of $\overline{\mathbf{Max}}(p_1p_2p_3 \cdots p_{n+1})$, $\overline{\mathsf{flip}}_k(\mathbf{x}) = \mathbf{q_t} \cdot_{n+1}$ appears before $\mathbf{y}$ in the listing as the first signed permutation of some signed bracelet sequence. If $\mathbf{t}$ is even, then $\mathbf{t}-1$ is odd and hence $\mathbf{q_{t-1}} \cdot p_{n+1}$ appears before $\mathbf{y}$ as a signed bracelet sequence representative. As noted earlier, the last signed permutation in such a bracelet sequence is $\mathbf{q_t} \cdot n+1$, and hence it appears before $\mathbf{y}$. Thus l is a greedy maximum-flip to go from $\mathbf{x}$ to $\mathbf{y}$ in this listing. ☐

Theorem 7.10. *For $n \geq 1$, the listing $\overline{\mathbf{Max}}(\mathbf{p})$ is a flip Gray code for signed permutations, where the first and last signed permutations differ by a flip of length 1.*

Proof. By applying Lemma 7.2, the length of the flip-sequence $\overline{\sigma'}_n$ is $2^n n! - 1$. Thus, since the flip-sequence for $\overline{\mathbf{Max}}(\mathbf{p})$ is $\overline{\sigma'}_n$ by Lemma 7.8, the number of signed permutations in the listing $\overline{\mathbf{Max}}(\mathbf{p})$ is $2^n n!$. Since the listing $\overline{\mathbf{Max}}(\mathbf{p})$ is equivalent to the greedy listing $\overline{\mathsf{MaxGreedy}}(\mathbf{p})$ by Lemma 7.9, each permutation in the listing must be unique. Thus, $\mathbf{Max}(\mathbf{p})$ is a signed permutation flip Gray code. Finally, from Lemma 7.7, the first and last signed permutation of the listing differ by a flip of length 1. ☐

An analysis on the average flip length, denoted $avg(n)$, required to generate $\overline{\mathbf{Max}}(\mathbf{p})$, follows a similar approach to the unsigned case. We consider the sequence to be circular to slightly simplify the analysis, and hence the average includes the final flip of length 1 to go from the last signed permutation to the first one. An upper bound on this average is obtained by bounding each flip that is less than or equal to $n-1$ by $n-1$. Since there are $\frac{2^n n!}{2}$ occurrences of n in $\overline{\sigma'}_n$, we obtain the following upper bound:

$$avg(n) \ \leq \ \frac{1}{2^n n!}\left(n \cdot \frac{2^n n!}{2} \ + \ (n-1) \cdot \frac{2^n n!}{2}\right) \ = \ n - \frac{1}{2}.$$

To obtain a lower bound for $n \geq 4$, we ignore all flips of length less than $n-2$. Observe that there are $\frac{2n-1}{2n} \cdot \frac{2^n n!}{2}$ occurrences of $n-1$ and $\frac{2(n-1)-1}{2(n-1)} \cdot \frac{2^{n-1}(n-1)!}{2}$ occurrences of $n-2$ in $\overline{\tau'}_n$ (and hence $\overline{\sigma'}_n$). Thus:

$$\begin{aligned}
avg(n) \ &\geq \ \frac{1}{2^n n!}\left(n \cdot \frac{2^n n!}{2} \ + \ (n-1) \cdot \frac{2n-1}{2n} \cdot \frac{2^n n!}{2} \ + \ (n-2) \cdot \frac{2(n-1)-1}{2(n-1)} \cdot \frac{2^{n-1}(n-1)!}{2}\right) \\
&= \ \frac{n}{2} + \frac{(n-1)(2n-1)}{4n} + \frac{(n-2)(2n-3)}{8n \cdot (n-1)} \\
&> \ \frac{n}{2} + \frac{2n^2-3n}{4n} + \frac{2n^2-7n}{8n^2} \\
&= \ \frac{n}{2} + (\frac{n}{2} - \frac{3}{4}) + (\frac{1}{4} - \frac{7}{8n}) \\
&= \ n - \frac{1}{2} - \frac{7}{8n}.
\end{aligned}$$

As n goes to infinity the average flip length approaches $n - \frac{1}{2}$.

8. Concluding Remarks

In this article we used the greedy approach from [18] to obtain exhaustive lists of (signed) permutations using (complemented) prefix-reversals. Each listing was reinterpreted using a recursive formulation which ultimately leads to efficient ranking, unranking, and generation algorithms for the listing [12]. More generally, this article provides a roadmap for turning simple greedy experiments into Gray codes that are suitable for applications.

While conducting our initial research for this article, we tested every possible prioritization of the flip operations with respect to the greedy algorithm. For permutations, we discovered two additional priorities that generate exhaustive lists for all n:

1. $\mathsf{PseudoMin}(\mathbf{p}) = \mathsf{Greedy}(\ \mathbb{P}(n), \mathbf{p}, \langle \mathsf{flip}_3, \mathsf{flip}_2, \mathsf{flip}_4, \ldots, \mathsf{flip}_n \rangle)$,
2. $\mathsf{PseudoMax}(\mathbf{p}) = \mathsf{Greedy}(\ \mathbb{P}(n), \mathbf{p}, \langle \mathsf{flip}_n, \mathsf{flip}_{n-1}, \ldots, \mathsf{flip}_4, \mathsf{flip}_2, \mathsf{flip}_3 \rangle)$.

The reader is encouraged to derive recurrences for these orders that are similar to $\mathbf{Min}(\mathbf{p})$ and $\mathbf{Max}(\mathbf{p})$ respectively, but with different base cases. In addition, we found one special case for $n \leq 10$,

$$\mathsf{Greedy}(\mathbb{P}(5), \mathbf{p}, \langle \mathsf{flip}_3, \mathsf{flip}_5, \mathsf{flip}_2, \mathsf{flip}_4 \rangle), \tag{5}$$

which generates all $5! = 120$ permutations. For signed permutations, we found no additional priorities that generate exhaustive lists for all n. However, we did find two special cases for $n \leq 7$,

$$\mathsf{Greedy}(\overline{\mathbb{P}}(3), \mathbf{p}, \langle \overline{\mathsf{flip}}_2, \overline{\mathsf{flip}}_3, \overline{\mathsf{flip}}_1 \rangle) \text{ and } \mathsf{Greedy}(\overline{\mathbb{P}}(3), \mathbf{p}, \langle \overline{\mathsf{flip}}_1, \overline{\mathsf{flip}}_3, \overline{\mathsf{flip}}_2 \rangle), \tag{6}$$

which generate all $3^2 3! = 48$ signed permutations. These results lead to the following conjectures.

Conjecture 8.1. *For $n > 5$, $\mathsf{MinGreedy}(\mathbf{p})$, $\mathsf{MaxGreedy}(\mathbf{p})$, $\mathsf{PseudoMin}(\mathbf{p})$, and $\mathsf{PseudoMax}(\mathbf{p})$ are the only greedy flip Gray codes for $\mathbb{P}(n)$.*

Conjecture 8.2. *For $n > 3$, $\overline{\mathsf{MinGreedy}}(\mathbf{p})$ and $\overline{\mathsf{MaxGreedy}}(\mathbf{p})$ are the only greedy flip Gray codes for $\overline{\mathbb{P}}(n)$.*

In other words, we conjecture that the minimum-flip and maximum-flip approaches are the only greedy algorithms for exhaustive pancake flipping, up to trivial modifications and small cases. The authors believe that settling Conjectures 8.1 and 8.2 could help lead to a deeper understanding of when the greedy approach to constructing Gray codes succeeds and fails.

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