

Inside the Binary Reflected Gray Code: Flip-Swap Languages in 2-Gray Code Order

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Abstract. A *flip-swap language* is a set S of binary strings of length n such that $S \cup \{0^n\}$ is closed under two operations (when applicable): (1) Flip the leftmost 1; and (2) Swap the leftmost 1 with the bit to its right. Flip-swap languages model many combinatorial objects including necklaces, Lyndon words, prefix normal words, left factors of k -ary Dyck words, and feasible solutions to 0-1 knapsack problems. We prove that any flip-swap language forms a cyclic 2-Gray code when listed in binary reflected Gray code (BRGC) order. Furthermore, a generic successor rule computes the next string when provided with a membership tester. The rule generates each string in the aforementioned flip-swap languages in $O(n)$ -amortized per string, except for prefix normal words of length n which require $O(n^{1.864})$ -amortized per string. Our work generalizes results on necklaces and Lyndon words by Vajnovski [Inf. Process. Lett. 106(3):96–99, 2008].

1 Introduction

Combinatorial generation studies the efficient generation of each instance of a combinatorial object, such as the $n!$ permutations of $\{1, 2, \dots, n\}$ or the $\frac{1}{n+1} \binom{2n}{n}$ binary trees with n nodes. The research area is fundamental to computer science and it has been covered by textbooks such as *Combinatorial Algorithms for Computers and Calculators* by Nijenhuis and Wilf [27], *Concrete Mathematics: A Foundation for Computer Science* by Graham, Knuth, and Patashnik [9], and *The Art of Computer Programming, Volume 4A, Combinatorial Algorithms* by Knuth [12]. In fact, Knuth’s section on *Generating Basic Combinatorial Patterns* is over 450 pages. The subject is important to every day programmers, and Arndt’s *Matters Computational: Ideas, Algorithms, Source Code* is an excellent practical resource [1]. A primary consideration is listing the instances of a combinatorial object so that consecutive instances differ by a specified *closeness condition*. Lists of this type are called *Gray codes*. This terminology is due to the eponymous *binary reflected Gray code (BRGC)* by Frank Gray, which orders the 2^n binary strings of length n so that consecutive strings differ in one bit. The BRGC was patented for a pulse code communication system in 1953 [10]. For example, the order for $n = 4$ is

$$\begin{aligned} &0000, 1000, 1100, 0100, 0110, 1110, 1010, 0010, \\ &0011, 1011, 1111, 0111, 0101, 1101, 1001, 0001. \end{aligned} \tag{1}$$

Variations that reverse the entire order or the individual strings are also commonly used in practice and in the literature. We note that the order in (1) is *cyclic* because the last and first strings also differ by the closeness condition, and this property holds for all n .

One challenge facing combinatorial generation is its relative surplus of breadth and lack of depth¹. For example, [1], [12], and [27] have separate subsections for different combinatorial objects, and the majority of the Gray codes are developed from first principles. Thus, it is important to encourage simple frameworks that can be applied to a variety of combinatorial objects. Previous work in this direction includes the following:

1. the ECO framework developed by Bacchelli, Barucci, Grazzini, and Pergola [2] that generates Gray codes for a variety of combinatorial objects such as Dyck words in constant amortized time per instance;
2. the twisted lexico computation tree by Takaoka [22] that generates Gray codes for multiple combinatorial objects in constant amortized time per instance;
3. loopless algorithms developed by Walsh [25] to generate Gray codes for multiple combinatorial objects, which extend algorithms initially given by Ehrlich in [8];
4. greedy algorithms observed by Williams [28] that provide a uniform understanding for many previous published results;
5. the reflectable language framework by Li and Sawada [13] for generating Gray codes of k -ary strings, restricted growth strings, and k -ary trees with n nodes;
6. the bubble language framework developed by Ruskey, Sawada and Williams [17] that provides algorithms to generate shift Gray codes for fixed-weight necklaces and Lyndon words, k -ary Dyck words, and representations of interval graphs;
7. the permutation language framework developed by Hartung, Hoang, Mütze and Williams [11] that provides algorithms to generate Gray codes for a variety of combinatorial objects based on encoding them as permutations.

We focus on an approach that is arguably simpler than all of the above: Start with a known Gray code and then *filter* or *induce* the list based on a subset of interest. In other words, the subset is listed in the relative order given by a larger Gray code, and the resulting order is a *sublist (Gray code)* with respect to it. Historically, the first sublist Gray code appears to be the *revolving door* Gray code for combinations [26]. A *combination* is a length n binary string with *weight* (i.e. number of ones) k . The Gray code is created by filtering the BRGC, as shown below for $n = 4$ and $k = 2$ (cf. (1))

$$\begin{aligned} &0000, 1000, 1100, 0100, 0110, 1110, 1010, 0010, \\ &0011, 1011, 1111, 0111, 0101, 1101, 1001, 0001. \end{aligned} \tag{2}$$

This order is a *transposition Gray code* as consecutive strings differ by transposing two bits². It can be generated *directly* (i.e. without filtering) by an efficient algorithm [26]. Transposition Gray codes are a special case of *2-Gray codes* where consecutive strings differ by flipping (i.e. complementing) at most two bits. Vajnovszki [23] proved that necklaces and Lyndon words form a cyclic 2-Gray code in BRGC order, and efficient algorithms can generate these sublist Gray codes directly [21]. Our goal is to expand upon the known languages that are 2-Gray codes in BRGC order, and which can be efficiently generated. To do this, we introduce a new class of languages.

¹ This is not to say that combinatorial generation is always easy. For example, the ‘middle levels’ conjecture was confirmed by Mütze [14] after 30 years and effort by hundreds of researchers.

² When each string is viewed as the incidence vector of a k -subset of $\{1, 2, \dots, n\}$, then consecutive k -subsets change via a “revolving door” (i.e. one value enters and one value exits).

A *flip-swap language* (with respect to 1) is a set \mathbf{S} of length n binary strings such that $\mathbf{S} \cup \{0^n\}$ is closed under two operations (when applicable): (1) Flip the leftmost 1; and (2) Swap the leftmost 1 with the bit to its right. A flip-swap language with respect to 0 is defined similarly. Flip-swap languages encode a wide variety of combinatorial objects.

Theorem 1. *The following sets of length n binary strings are flip-swap languages:*

Flip-Swap languages (with respect to 1)	Flip-Swap languages (with respect to 0)
i. all strings	▷ all strings
ii. strings with weight $\leq k$	▷ strings with weight $\geq k$
iii. strings $\leq \gamma$	▷ strings $\geq \gamma$
iv. strings with $\leq k$ inversions re: 0^*1^*	▷ strings with $\leq k$ inversions re: 1^*0^*
v. strings with $\leq k$ transpositions re: 0^*1^*	▷ strings with $\leq k$ transpositions re: 1^*0^*
vi. strings $<$ their reversal	▷ strings $>$ their reversal
vii. strings \leq their reversal (neckties)	▷ strings \geq their reversal
viii. strings $<$ their complemented reversal	▷ strings $>$ their complemented reversal
ix. strings \leq their complemented reversal	▷ strings \geq their complemented reversal
x. strings with forbidden 10^t	▷ strings with forbidden 01^t
xi. strings with forbidden prefix 1γ	▷ strings with forbidden prefix 0γ
xii. 0-prefix normal words	▷ 1-prefix normal words
xiii. necklaces (smallest rotation)	▷ necklaces (largest rotation)
xiv. Lyndon words	▷ aperiodic necklaces (largest rotation)
xv. prenecklaces (smallest rotation)	▷ prenecklaces (largest rotation)
xvi. pseudo-necklaces with respect to 0^*1^*	▷ pseudo-necklaces with respect to 1^*0^*
xvii. left factors of k -ary Dyck words	
xviii. feasible solutions to 0-1 knapsack problems	

Our second result is that every flip-swap language forms a cyclic 2-Gray code when listed in BRGC order. This generalizes the previous sublist BRGC results [21, 23].

Theorem 2. *When a flip-swap language \mathbf{S} is listed in BRGC order the resulting listing is a 2-Gray code. If \mathbf{S} includes 0^n then the listing is cyclic.*

Our third result is a generic successor rule, which efficiently computes the next string in the 2-Gray code of a flip-swap language, so long as a fast membership test is given.

Theorem 3. *The languages in Theorem 1 can be generated in $O(n)$ -amortized time per string, with the exception of prefix normal words which require $O(n^{1.864})$ -time.*

In Section 2, we formally define our version of the BRGC. In Section 3, we prove Theorem 1, and define the flip-swap partially ordered set. In Section 4, we give our generic successor rule and prove Theorem 2. In Section 5, we present a generic generation algorithm that list out each string of a flip-swap language, and we prove Theorem 3.

2 The Binary Reflected Gray Code

Let $\mathbf{B}(n)$ denote the set of length n binary strings. Let $BRGC(n)$ denote the listing of $\mathbf{B}(n)$ in BRGC order. Let $\overline{BRGC}(n)$ denote the listing $BRGC(n)$ in reverse order.

$n = 4$	all	necklaces	0-PNW	≤ 1001	$k \leq 2$	neckties	
BRGC	i.	xiii.	xii.	iii.	ii.	vii.	
0000	✓	✓	✓	✓	✓	✓	
1000	✓			✓	✓		
1100	✓			✓	✓		
0100	✓			✓	✓		
0110	✓		✓	✓	✓		
1110	✓			✓	✓		
1010	✓			✓	✓		
0010	✓		✓	✓	✓		
0011	✓	✓	✓	✓	✓		
1011	✓			✓	✓		
1111	✓	✓					
0111	✓	✓	✓	✓			
0101	✓	✓	✓	✓	✓		
1101	✓			✓	✓		
1001	✓			✓	✓		
0001	✓	✓	✓	✓	✓		

(a) String membership in 6 flip-swap languages. (b) Visualizing the 2-Gray codes in (a).

Table 1: Flip-swap languages ordered as sublists of the binary reflected Gray code. Theorem 1 covers each language, so the resulting orders are 2-Gray codes.

Then $BRGC(n)$ can be defined recursively as follows, where $\mathcal{L} \cdot x$ denotes the listing \mathcal{L} with the character x appended to the end of each string:

$$BRGC(n) = \begin{cases} 0, 1 & \text{if } n = 1; \\ BRGC(n-1) \cdot 0, \overline{BRGC(n-1)} \cdot 1 & \text{if } n > 1. \end{cases}$$

For example, $BRGC(2) = 00, 10, 11, 01$ and $\overline{BRGC}(2) = 01, 11, 10, 00$, thus

$$BRGC(3) = 000, 100, 110, 010, 011, 111, 101, 001.$$

This definition of BRGC order is the same as the one used by Vajnovzski [23]. When the strings are read from right-to-left, we obtain the classic definition of BRGC order [10]. For flip-swap languages with respect to 0, we interchange the roles of the 0s and 1s; however, for our discussions we will focus on flip-swap languages with respect to 1. Table 1 illustrates $BRGC(4)$ and six flip-swap languages listed in Theorem 1.

3 Flip-swap languages

In this section, we formalize some of the non-obvious flip-swap languages stated in Theorem 1. Then we prove Theorem 1 for a subset of the listed languages including necklaces, prefix normal words, and feasible solutions to the 0-1 knapsack problems. The remainder of the languages are proved in the Appendix.

Consider a binary string $\alpha = b_1 b_2 \dots b_n$. The *weight* of α is the number of 1s it contains. An *inversion* in α with respect to 0^*1^* is an index pair (i, j) such that $i < j$ and $b_i = 1$ and $b_j = 0$. The number of *transpositions* of α with respect to another

binary string β of length n is the minimum number of adjacent transpositions required to transform α to β .

A *necklace* is the lexicographically smallest (largest) string in an equivalence class under rotation. An *aperiodic necklace* is a necklace that cannot be written in the form β^j for some $j < n$. A *Lyndon word* is an aperiodic necklace when using the lexicographically smallest string as the representative. A *prenecklace* is a prefix of a necklace. A *block* with respect to 0^*1^* is a maximal substring of the form 0^*1^* . A string $\alpha = b_1b_2 \cdots b_n = B_bB_{b-1} \cdots B_1$ is a *pseudo-necklace* with respect to 0^*1^* if $B_b \leq B_i$ for all $1 \leq i < b$.

A *k-ary Dyck word* is a binary string of length $n = tk$ with t copies of 1 and $t(k-1)$ copies of 0 such that every prefix has at most $k-1$ copies of 0 for every 1. The set of length n prefixes of *k-ary Dyck words* is called *left factors of k-ary Dyck words*.

Let $flip_\alpha(i)$ be the string obtained by complementing b_i . Let $swap_\alpha(i, j)$ be the string obtained by swapping b_i and b_j . When the context is clear we use $flip(i)$ and $swap(i, j)$ instead of $flip_\alpha(i)$ and $swap_\alpha(i, j)$. Also, let $\ell_0(\alpha)$ denote the position of the leftmost 0 of α or $n+1$ if no such position exists. Similarly, let $\ell_1(\alpha)$ denote the position of the leftmost 1 of α or $n+1$ if no such position exists. We now prove that binary strings, necklaces, prefix normal words, and feasible solutions to the 0-1 knapsack problems are flip-swap languages with respect to 1.

Binary strings: Obviously the set $\mathbf{B}(n)$ satisfies the two closure properties of a flip-swap language and thus is a flip-swap language. In fact, the BRGC order induces a cyclic 1-Gray code for $\mathbf{B}(n)$ [12, 15].

Necklaces: Let $\mathbf{N}(n)$ be the set of necklaces of length n and $\alpha = 0^j1b_{j+2}b_{j+3} \cdots b_n$ be a necklace in $\mathbf{N}(n)$. By the definition of necklace, it is easy to see that $flip_\alpha(\ell_\alpha) = 0^{j+1}b_{j+2}b_{j+3} \cdots b_n \in \mathbf{N}(n)$ and thus $\mathbf{N}(n)$ satisfies the flip-first property. For the swap-first operation, observe that if $\alpha \neq 0^{n-1}1$ and $b_{j+2} = 1$, then the swap-first operation produces the same necklace. Otherwise if $\alpha \neq 0^{n-1}1$ and $b_{j+2} = 0$, then the swap-first operation produces the string $0^{j+1}1b_{j+3}b_{j+4} \cdots b_n$ which is clearly a necklace. Thus, the set of necklaces is a flip-swap language.

Prefix normal words: A binary string α is *prefix normal* with respect to 0 (also known as 0-prefix normal word) if no substring of α has more 0s than its prefix of the same length. For example, the string 001010010111011 is a 0-prefix normal word but the string 001010010011011 is not because it has a substring of length 5 with four 0s while the prefix of length 5 has only three 0s.

Observe that the set of 0-prefix normal words of length n satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations either increases or maintain the number of 0s in its prefix. Thus, the set of 0-prefix normal words of length n is a flip-swap language.

Feasible solutions to 0-1 knapsack problems: The input to a 0-1 *knapsack problem* is a knapsack capacity W , and a set of n items each of which has a non-negative weight $w_i \geq 0$ and a value v_i . A subset of items is *feasible* if the total weight of the items in the subset is less than or equal to the capacity W . Typically, the goal of the problem is to find a feasible subset with the maximum value, or to decide if a feasible subset exists with value $\geq c$.

Given the input to a 0-1 knapsack problem, we reorder the items by non-decreasing weight. That is, $w_i \geq w_{i+1}$ for $1 \leq i \leq n - 1$. Notice that the incidence vectors of feasible subsets are now a flip-swap language. More specifically, flipping any 1 to 0 causes the subset sum to decrease, and so does swapping any 1 with the bit to its right. Hence, the language satisfies the flip-first and the swap-first closure properties and is a flip-swap language.

3.1 Flip-Swap poset

In this section we introduce a poset whose ideals correspond to a flip-swap language which includes the string 0^n .

Let $\alpha = b_1 b_2 \cdots b_n$ be a length n binary string. We define $\tau(\alpha)$ as follows:

$$\tau(\alpha) = \begin{cases} \alpha & \text{if } \alpha = 0^n, \\ \text{flip}_\alpha(\ell_\alpha) & \text{if } \alpha \neq 0^n \text{ and } (\ell_\alpha = n \text{ or } b_{\ell_\alpha+1} = 1) \quad (\text{flip-first}), \\ \text{swap}_\alpha(\ell_\alpha, \ell_\alpha + 1) & \text{otherwise} \quad (\text{swap-first}). \end{cases}$$

Let $\tau^t(\alpha)$ denote the string that results from applying the τ operation t times to α . We define the binary relation $<_R$ on $\mathbf{B}(n)$ to be the transitive closure of the cover relation τ , that is $\beta <_R \alpha$ if $\beta \neq \alpha$ and $\beta = \tau^t(\alpha)$ for some $t > 0$. It is easy to see that the binary relation $<_R$ is irreflexive, anti-symmetric and transitive. Thus $<_R$ is a strict partial order. The relation $<_R$ on binary strings defines our flip-swap poset.

Definition 1. *The flip-swap poset $\mathcal{P}(n)$ is a strict poset with $\mathbf{B}(n)$ as the ground set and $<_R$ as the strict partial order.*

Figure 1 shows the Hasse diagram of $\mathcal{P}(4)$ with the ideal for binary strings of length 4 that are lexicographically smaller or equal to 1001 in bold. Observe that $\mathcal{P}(n)$ is always a tree with 0^n as the unique minimum element, and that its ideals are the subtrees that contain this minimum.

Lemma 1. *A set \mathbf{S} over $\mathbf{B}(n)$ that includes 0^n is a flip-swap language if and only if \mathbf{S} is an ideal of $\mathcal{P}(n)$.*

Proof. Let \mathbf{S} be a flip-swap language over $\mathbf{B}(n)$ and α be a string in \mathbf{S} . Since \mathbf{S} is a flip-swap language, \mathbf{S} satisfies the flip-first and swap-first properties and thus $\tau(\alpha)$ is a string in \mathbf{S} . Therefore every string $\gamma <_R \alpha$ is in \mathbf{S} and hence \mathbf{S} is an ideal of $\mathcal{P}(n)$. The other direction is similar. \square

If \mathbf{S} is a set of binary strings and γ is a binary string, then the *quotient* of \mathbf{S} and γ is $\mathbf{S}/\gamma = \{\alpha \mid \alpha\gamma \in \mathbf{S}\}$.

Lemma 2. *If \mathbf{S}_1 and \mathbf{S}_2 are flip-swap languages and γ is a binary string, then $\mathbf{S}_1 \cap \mathbf{S}_2$, $\mathbf{S}_1 \cup \mathbf{S}_2$ and \mathbf{S}_1/γ are flip-swap languages.*

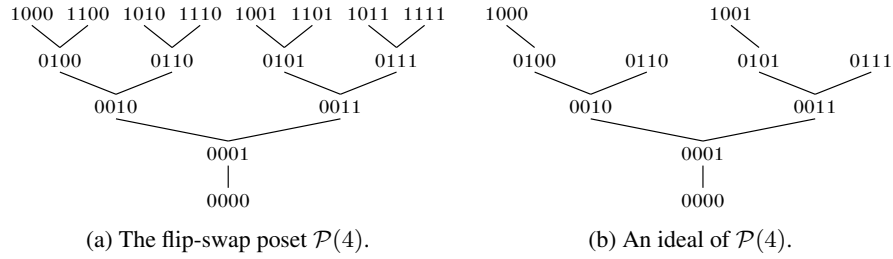


Fig. 1: Flip-swap languages are the ideals of the flip-swap poset. The ideal in (b) contains the 4-bit binary strings that are ≤ 1001 with respect to lexicographic order.

Proof. Let \mathbf{S}_1 and \mathbf{S}_2 be two flip-swap languages and let γ be a binary string. The intersection and union of ideals of any poset are also ideals of that poset, so $\mathbf{S}_1 \cap \mathbf{S}_2$ and $\mathbf{S}_1 \cup \mathbf{S}_2$ are flip-swap languages. Now consider $\alpha \in \mathbf{S}_1/\gamma$.

Suppose $\alpha \in \mathbf{S}_1/\gamma$ for some non-empty γ where $j = |\alpha|$. This means that $\alpha\gamma \in \mathbf{S}_1$. Consider three cases depending $l_{\alpha\gamma}$. If $l_{\alpha\gamma} < j$, then clearly $\tau(\alpha\gamma) = \tau(\alpha)\gamma$. From Lemma 1, $\tau(\alpha)\gamma \in \mathbf{S}_1$ and thus $\tau(\alpha) \in \mathbf{S}_1/\gamma$. If $l_{\alpha\gamma} = j$, then $\alpha = 0^{j-1}1$ and $\tau(\alpha) = 0^j$. Since \mathbf{S}_1 is a flip-swap language $0^j\gamma \in \mathbf{S}_1$. Again this implies that $\tau(\alpha) \in \mathbf{S}_1/\gamma$. If $l_{\alpha\gamma} > j$ then $\alpha = 0^j$ and $\tau(\alpha) = \alpha$ in this case. For each case we have shown that $\tau(\alpha) \in \mathbf{S}_1/\gamma$ and thus \mathbf{S}_1/γ is a flip-swap language by Lemma 1. \square

Corollary 1. *Flip-swap languages are closed under union, intersection, and quotient.*

Proof. Let \mathbf{S}_A and \mathbf{S}_B be flip-swap languages and γ be a binary string. Since \mathbf{S}_A and \mathbf{S}_B can be represented by ideals of the flip-swap poset, possibly excluding 0^n , by Lemma 2 the sets $\mathbf{S}_A \cap \mathbf{S}_B$, $\mathbf{S}_A \cup \mathbf{S}_B$ and \mathbf{S}_A/γ are flip-swap languages. \square

Lemma 3. *If $\alpha\gamma$ is a binary string in a flip-swap language \mathbf{S} , then $0^{|\alpha|}\gamma \in \mathbf{S}$.*

Proof. This result follows from the flip-first property of flip-swap languages. \square

4 A generic successor rule for flip-swap languages

Consider any flip-swap language \mathbf{S} that includes the string 0^n . Let $\mathcal{BRGC}(\mathbf{S})$ denote the listing of \mathbf{S} in BRGC order. Given a string $\alpha \in \mathbf{S}$, we define a generic *successor rule* that computes the string following α in the cyclic listing $\mathcal{BRGC}(\mathbf{S})$.

Let $\alpha = b_1b_2 \cdots b_n$ be a string in \mathbf{S} . Let t_α be the leftmost position such that $flip_\alpha(t_\alpha) \in \mathbf{S}$ when $|\mathbf{S}| > 1$, such a t_α exists since \mathbf{S} satisfies the flip-first property and $|\mathbf{S}| > 1$. Recall that l_α is defined to be the position of the leftmost 1 of α (or $|\alpha| + 1$ if no such position exists). Notice that $t_\alpha \leq l_\alpha$ when $|\mathbf{S}| > 1$ since \mathbf{S} is a flip-swap language.

Let $flip2_\alpha(i, j)$ be the string obtained by complementing both b_i and b_j . When the context is clear we use $flip2(i, j)$ instead of $flip2_\alpha(i, j)$. Also, let $w(\alpha)$ denote the

Necklaces	Parity of $w(\alpha)$	t_α	ℓ_α	Successor	Case
000000	even	6		$flip2(5, 6)$	(4c)
000011	even	3		$flip2(2, 3)$	(4c)
011011	even	2		$flip(2)$	(4b)
001011	odd		3	$flip(4)$	(4e)
001111	even	2		$flip2(1, 2)$	(4c)
111111	even	1		$flip(1)$	(4b)
011111	odd		2	$flip(3)$	(4e)
010111	even	3		$flip(2)$	(4b)
000111	odd		4	$flip(5)$	(4e)
000101	even	2		$flip(2)$	(4b)
010101	odd		2	$flip2(2, 3)$	(4d)
001101	odd		3	$flip(4)$	(4e)
001001	even	3		$flip(3)$	(4b)
000001	odd			$flip(6)$	(4a)

Table 2: The necklaces of length 6 induced by successive applications the function f starting from 000000. The sixth column of the table lists out the corresponding rules in f that apply to each necklace to obtain the next necklace.

number of 1s of α . We claim that the following function f computes the next string in the cyclic ordering $BRGC(\mathbf{S})$:

$$f(\alpha) = \begin{cases} 0^n & \text{if } \alpha = 0^{n-1}1; & (4a) \\ flip_\alpha(t_\alpha) & \text{if } w(\alpha) \text{ is even and } (t_\alpha = 1 \text{ or } flip2_\alpha(t_\alpha - 1, t_\alpha) \notin \mathbf{S}); & (4b) \\ flip2_\alpha(t_\alpha - 1, t_\alpha) & \text{if } w(\alpha) \text{ is even and } flip2_\alpha(t_\alpha - 1, t_\alpha) \in \mathbf{S}; & (4c) \\ flip2_\alpha(\ell_\alpha, \ell_\alpha + 1) & \text{if } w(\alpha) \text{ is odd and } flip_\alpha(\ell_\alpha + 1) \notin \mathbf{S}; & (4d) \\ flip_\alpha(\ell_\alpha + 1) & \text{if } w(\alpha) \text{ is odd and } flip_\alpha(\ell_\alpha + 1) \in \mathbf{S}. & (4e) \end{cases}$$

Thus, successive applications of the function f on a flip-swap language \mathbf{S} , starting with the string 0^n , list out each string in \mathbf{S} in BRGC order. As an illustration of the function f , successive applications of this rule for the set of necklaces of length 6 starting with the necklace 000000 produce the listing in Table 2.

Theorem 4. *If \mathbf{S} is a flip-swap language including the string 0^n and $|\mathbf{S}| > 1$, then $f(\alpha)$ is the string immediately following the string α in \mathbf{S} in the cyclic ordering $BRGC(\mathbf{S})$.*

We will provide a detailed proof of this theorem in the next subsection. Observe that each rule in f complements at most two bits and thus successive strings in \mathbf{S} differ by at most two bit positions. Observe that when 0^n is excluded from \mathbf{S} , then $BRGC(\mathbf{S})$ is still a 2-Gray code (although not necessarily cyclic). This proves Theorem 2.

4.1 Proof of Theorem 4

This section proves Theorem 4. We begin with a lemma by Vajnovszki [23], and a remark that is due to the fact that $0^{n-1}1$ is in a flip-swap language \mathbf{S} when $|\mathbf{S}| > 1$.

Lemma 4. Let $\alpha = b_1b_2 \cdots b_n$ and β be length n binary strings such that $\alpha \neq \beta$. Let r be the rightmost position in which α and β differ. Then α comes before β in BRGC order (denoted by $\alpha \prec \beta$) if and only if $w(b_rb_{r+1} \cdots b_n)$ is even.

Remark 1. A flip-swap language \mathbf{S} in BRGC order ends with $0^{n-1}1$ when $|\mathbf{S}| > 1$.

Let $\text{succ}(\mathbf{S}, \alpha)$ be the successor of α in \mathbf{S} in BRGC order (i.e. the string after α in the cyclic ordering $\mathcal{BRGC}(\mathbf{S})$). Next we provide two lemmas, and then prove Theorem 4.

Lemma 5. Let \mathbf{S} be a flip-swap language with $|\mathbf{S}| > 1$ and α be a string in \mathbf{S} . Let t_α be the leftmost position such that $\text{flip}_\alpha(t_\alpha) \in \mathbf{S}$. If $w(\alpha)$ is even, then t_α is the rightmost position in which α and $\text{succ}(\mathbf{S}, \alpha)$ differ.

Proof. By contradiction. Let $\alpha = b_1b_2 \cdots b_n$ and $\beta = \text{succ}(\mathbf{S}, \alpha)$. Let r be the rightmost position in which α and β differ with $r \neq t_\alpha$. If $t_\alpha > r$, then β has the suffix $1b_{r+1}b_{r+2} \cdots b_n$ since $b_r = 0$ because $r < t_\alpha \leq \ell_\alpha$. Thus by the flip-first property, $0^{r-1}1b_{r+1}b_{r+2} = \text{flip}_\alpha(r) \in \mathbf{S}$ and $r < t_\alpha$, a contradiction.

Otherwise if $t_\alpha < r$, then let $\gamma = \text{flip}_\alpha(t_\alpha)$. Clearly $\gamma \neq \alpha$. Now observe that $w(b_t b_{t+1} \cdots b_n)$ is even because $t_\alpha \leq \ell_\alpha$ and $w(\alpha)$ is even, and thus by Lemma 4, $\alpha \prec \gamma$. Also, γ has the suffix $b_r b_{r+1} \cdots b_n$ and $w(b_r b_{r+1} \cdots b_n)$ is even because $\alpha \prec \beta$ and r is the rightmost position α and β differ, and thus also by Lemma 4, $\gamma \prec \beta$. Thus $\alpha \prec \gamma \prec \beta$, a contradiction. Therefore $r = t_\alpha$. \square

Lemma 6. Let \mathbf{S} be a flip-swap language with $|\mathbf{S}| > 1$ and $\alpha \neq 0^{n-1}1$ be a string in \mathbf{S} . If $w(\alpha)$ is odd, then $\ell_\alpha + 1$ is the rightmost position in which α and $\text{succ}(\mathbf{S}, \alpha)$ differ.

Proof. Since $\alpha \neq 0^{n-1}1$ and $w(\alpha)$ is odd, $\ell_\alpha < n - 1$. We now prove the lemma by contradiction. Let $\alpha = b_1b_2 \cdots b_n$ and $\beta = \text{succ}(\mathbf{S}, \alpha)$. Let $r \neq \ell_\alpha + 1$ be the rightmost position in which α and β differ. If $r < \ell_\alpha + 1$, then $w(b_r b_{r+1} \cdots b_n)$ is odd but $\alpha \prec \beta$, a contradiction by Lemma 4. Otherwise if $r > \ell_\alpha + 1$, then let $\gamma = \text{flip}_{2_\alpha}(\ell_\alpha, \ell_\alpha + 1)$. Clearly $\gamma \neq \alpha$, and by the flip-first and swap-first properties, $\gamma \in \mathbf{S}$. Also, observe that $w(b_{\ell_\alpha+1} b_{\ell_\alpha+2} \cdots b_n)$ is even because $w(\alpha)$ is odd, and thus by Lemma 4, $\alpha \prec \gamma$. Further, γ has the suffix $b_r b_{r+1} \cdots b_n$ and $w(b_r b_{r+1} \cdots b_n)$ is even because $\alpha \prec \beta$ and r is the rightmost position α and β differ, and thus also by Lemma 4, $\gamma \prec \beta$. Thus $\alpha \prec \gamma \prec \beta$, a contradiction. Therefore $r = \ell_\alpha + 1$. \square

Proof of Theorem 4. Let $\alpha = a_1a_2 \cdots a_n$ and $\beta = \text{succ}(\mathbf{S}, \alpha) = b_1b_2 \cdots b_n$. Let t_α be the leftmost position such that $\text{flip}_\alpha(t_\alpha) \in \mathbf{S}$. First we consider the case when $\alpha = 0^{n-1}1$. Recall that the first string in $\mathbf{B}(n)$ in BRGC order is 0^n [15] and 0^n is a string in \mathbf{S} by Lemma 3. Also, the last string in \mathbf{S} in BRGC order is $0^{n-1}1$ by Remark 1 when $|\mathbf{S}| > 1$. Thus the string that appears immediately after α in the cyclic ordering $\mathcal{BRGC}(\mathbf{S})$ is $f(\alpha)$ when $\alpha = 0^{n-1}1$. In the remainder of the proof, $\alpha \neq 0^{n-1}1$ and we consider the following two cases.

Case 1: $w(\alpha)$ is even: If $t_\alpha = 1$, then clearly $\beta = \text{flip}_\alpha(t_\alpha) = f(\alpha)$. For the remainder of the proof, $t_\alpha > 1$.

Since $t_\alpha \leq \ell_\alpha$, $\text{flip}2_\alpha(t_\alpha - 1, t_\alpha)$ has the prefix $0^{t_\alpha-2}1$. We now consider the following two cases. If $\text{flip}2_\alpha(t_\alpha - 1, t_\alpha) \notin \mathbf{S}$, then $\text{flip}_\alpha(t_\alpha)$ is the only string in \mathbf{S} that has t_α as the rightmost position that differ with α and has the prefix $0^{t_\alpha-2}$. Therefore, $\beta = \text{flip}_\alpha(t_\alpha) = f(\alpha)$. Otherwise, $\text{flip}2_\alpha(t_\alpha - 1, t_\alpha)$ and $\text{flip}_\alpha(t_\alpha)$ are the only strings in \mathbf{S} that have t_α as the rightmost position that differ with α and have the prefix $0^{t_\alpha-2}$. By Lemma 4, $\text{flip}2_\alpha(t_\alpha - 1, t_\alpha) \prec \text{flip}_\alpha(t_\alpha)$ since $w(1\bar{a}_{t_\alpha}a_{t_\alpha+1}a_{t_\alpha+2}\cdots a_n)$ is even. Thus, $\beta = \text{flip}2_\alpha(t_\alpha - 1, t_\alpha) = f(\alpha)$.

Case 2: $w(\alpha)$ is odd: By Lemma 6, β has the suffix $\bar{a}_{\ell_\alpha+1}a_{\ell_\alpha+2}a_{\ell_\alpha+3}\cdots a_n$. If $\text{flip}_\alpha(\ell_\alpha + 1) \notin \mathbf{S}$, then by the flip-first and swap-first properties, $\text{flip}2_\alpha(\ell_\alpha, \ell_\alpha + 1)$ is the only string in \mathbf{S} that has $\ell_\alpha + 1$ as the rightmost position that differ with β . Thus, $\beta = \text{flip}2_\alpha(\ell_\alpha, \ell_\alpha + 1) = f(\alpha)$. Otherwise by Lemma 4, any string $\gamma \in \mathbf{S}$ with the suffix $\bar{a}_{\ell_\alpha+1}a_{\ell_\alpha+2}a_{\ell_\alpha+3}\cdots a_n$ and $\gamma \neq \text{flip}_\alpha(\ell_\alpha + 1)$ has $\text{flip}_\alpha(\ell_\alpha + 1) \prec \gamma$ because $w(1\bar{a}_{\ell_\alpha+1}a_{\ell_\alpha+2}a_{\ell_\alpha+3}\cdots a_n)$ is even. Thus, $\beta = \text{flip}_\alpha(\ell_\alpha + 1) = f(\alpha)$.

Therefore, the string immediately after α in the cyclic ordering $\text{BRGC}(\mathbf{S})$ is $f(\alpha)$. \square

5 Generation algorithm for flip-swap languages

In this section we present a generic algorithm to generate 2-Gray codes for flip-swap languages via the function f .

A naïve approach to implement f is to find t_α by test flipping each bit in α to see if the result is also in the set when $w(\alpha)$ is even; or test flipping the $(\ell_\alpha + 1)$ -th bit of α to see if the result is also in the set when $w(\alpha)$ is odd. Since $t_\alpha \leq \ell_\alpha$, we only need to examine the length $\ell_\alpha - 1$ prefix of α to find t_α . Such a test can be done in $O(nm)$ time, where $O(m)$ is the time required to complete the membership test of the set under consideration. Pseudocode of the function f is given in Algorithm 1.

To list out each string of a flip-swap language \mathbf{S} in BRGC order, we can repeatedly apply the function f until it reaches the starting string. We also maintain $w(\alpha)$ and ℓ_α which can be easily maintained in $O(n)$ time for each string generated. We also add a condition to avoid printing the string 0^n if 0^n is not a string in \mathbf{S} . Pseudocode for this algorithm, starting with the string 0^n , is given in Algorithm 2. The algorithm can easily be modified to generate the corresponding counterpart of \mathbf{S} with respect to 0.

A simple analysis shows that the algorithm generates \mathbf{S} in $O(nm)$ -time per string. A more thorough analysis improves this to $O(n + m)$ -amortized time per string.

Theorem 5. *If \mathbf{S} is a flip-swap language, then the algorithm BRGC produces $\text{BRGC}(\mathbf{S})$ in $O(n + m)$ -amortized time per string, where $O(m)$ is the time required to complete the membership tester for \mathbf{S} .*

Proof. Let $\alpha = a_1a_2\cdots a_n$ be a string in \mathbf{S} . Clearly f can be computed in $O(n)$ time when $w(\alpha)$ is odd. Otherwise when $w(\alpha)$ is even, the **while** loop in line 5 of Algorithm 1 performs a membership tester on each string $\beta = b_1b_2\cdots b_n$ in \mathbf{S} with $b_{\ell_\alpha}b_{\ell_\alpha+1}\cdots b_n = a_{\ell_\alpha}a_{\ell_\alpha+1}\cdots a_n$ and $w(b_1b_2\cdots b_{\ell_\alpha-1}) = 1$. Observe that each of these strings can only be examined by the membership tester once, or otherwise the

Algorithm 1 Pseudocode of the implementation of the function f .

```

1: function  $f(\alpha)$ 
2:   if  $\alpha = 0^{n-1}1$  then  $flip_\alpha(n)$ 
3:   else if  $w(\alpha)$  is even then
4:      $t_\alpha \leftarrow \ell_\alpha$ 
5:     while  $t_\alpha > 1$  and  $flip_\alpha(t_\alpha - 1) \in \mathbf{S}$  do  $t_\alpha \leftarrow t_\alpha - 1$ 
6:     if  $t_\alpha \neq 1$  and  $flip_{2_\alpha}(t_\alpha - 1, t_\alpha) \in \mathbf{S}$  then  $\alpha \leftarrow flip_{2_\alpha}(t_\alpha - 1, t_\alpha)$ 
7:     else  $\alpha \leftarrow flip_\alpha(t_\alpha)$ 
8:   else
9:     if  $flip_\alpha(\ell_\alpha + 1) \notin \mathbf{S}$  then  $\alpha \leftarrow flip_{2_\alpha}(\ell_\alpha, \ell_\alpha + 1)$ 
10:    else  $\alpha \leftarrow flip_\alpha(\ell_\alpha + 1)$ 

```

Algorithm 2 Algorithm to list out each string of a flip-swap language \mathbf{S} in BRGC order.

```

1: procedure  $BRGC$ 
2:    $\alpha = b_1b_2 \cdots b_n \leftarrow 0^n$ 
3:   do
4:     if  $\alpha \neq 0^n$  or  $0^n \in \mathbf{S}$  then  $Print(\alpha)$ 
5:      $f(\alpha)$ 
6:      $w(\alpha) \leftarrow 0$ 
7:     for  $i$  from  $n$  down to 1 do
8:       if  $b_i = 1$  then  $w(\alpha) \leftarrow w(\alpha) + 1$ 
9:       if  $b_i = 1$  then  $\ell_\alpha \leftarrow i$ 
10:  while  $\alpha \neq 0^n$ 

```

while loop in line 5 of Algorithm 1 produces the same t_α which results in a duplicated string, a contradiction. Thus, the total number of membership testers performed by the algorithm is bound by $|\mathbf{S}|$, and therefore f runs in $O(m)$ -amortized time per string. Finally, since the other part of the algorithm runs in $O(n)$ time per string, the algorithm $BRGC$ runs in $O(n + m)$ -amortized time per string. \square

The membership tests in this paper can be implemented in $O(n)$ time and $O(n)$ space; see [3, 7, 20] for necklaces, Lyndon words, prenecklaces and pseudo-necklaces of length n . One exception is the test for prefix normal words of length n , which requires $O(n^{1.864})$ time and $O(n)$ space [5]. Together with the above theorem, this proves Theorem 3.

Visit the Combinatorial Object Server [6] for a C implementation of our algorithms.

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Appendix: Proofs for flip-swap languages

This section provides the proofs for the remainder of the languages in Theorem 1. Unless otherwise stated, the discussion of flip-swap languages are with respect to 1.

Binary strings with weight $\leq k$

Recall the weight of a binary string is the number of 1s it contains. Let \mathbf{S} be the set of binary strings of length n having weight less than or equal to some k . Observe that \mathbf{S} satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations either decrease or maintain the weight. Thus, \mathbf{S} is a flip-swap language.

Binary strings $\leq \gamma$

Let \mathbf{S} be the set of binary strings of length n with each string lexicographically smaller or equal to some string γ . Observe that \mathbf{S} satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations either make the resulting string lexicographically smaller or produce the same string. Thus, \mathbf{S} is a flip-swap language.

Binary strings with $\leq k$ inversions

Recall that an *inversion* with respect to 0^*1^* in a binary string $\alpha = b_1b_2 \cdots b_n$ is any $b_i = 1$ and $b_j = 0$ such that $i < j$. For example when $\alpha = 100101$, it has 4 inversions: (b_1, b_2) , (b_1, b_3) , (b_1, b_5) , (b_4, b_5) . Let \mathbf{S} be the set of binary strings of length n with less than or equal to k inversions with respect to 0^*1^* . Observe that \mathbf{S} satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations either decrease or maintain the number of inversions. Thus, \mathbf{S} is a flip-swap language.

Binary strings with $\leq k$ transpositions

Recall that the number of *transpositions* of a binary string $\alpha = b_1b_2 \cdots b_n$ with respect to 0^*1^* is the minimum number of $swap(i, j)$ operations required to change α into the form 0^*1^* . For example, the number of transpositions of the string 100101 is 1. Let \mathbf{S} be the set of binary strings of length n with less than or equal to k transpositions with respect to 0^*1^* . Observe that \mathbf{S} satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations either decrease or maintain the number of transpositions. Thus, \mathbf{S} is a flip-swap language.

Binary strings $< \text{or } \leq$ their reversal

Let \mathbf{S} be the set of binary strings of length n with each string lexicographically smaller than their reversal. Observe that \mathbf{S} satisfies the swap-first property as the swap-first operation either produces the same string, or makes the resulting string lexicographically

smaller while its reversal lexicographically larger. Furthermore, $\mathbf{S} \cup \{0^n\}$ satisfies the flip-first property as the flip-first operation complements the most significant bit of α but the least significant bit of its reversal when $w(\alpha) > 1$; or otherwise produces the string 0^n when $w(\alpha) = 1$. Thus, \mathbf{S} is a flip-swap language. The proof for the set of binary strings of length n with each string lexicographically smaller than or equal to their reversal is similar to the proof for \mathbf{S} .

Equivalence class of strings under reversal has also been called neckties [19].

Binary strings $<$ or \leq their complemented reversal

Let \mathbf{S} be the set of binary strings of length n with each string lexicographically smaller than (or equal to) its complemented reversal. Observe that \mathbf{S} satisfies the flip-first property as the flip-first operation makes the resulting string lexicographically smaller while its complemented reversal lexicographically larger. Furthermore, \mathbf{S} satisfies the swap-first property as the swap-first operation either produces the same string, or complements the most significant bit of α and also a 1 of its complemented reversal. Thus, the resulting string must also be less than its complemented reversal. Thus, \mathbf{S} is a flip-swap language.

Binary strings with forbidden 10^t

Let \mathbf{S} be the set of binary strings of length n without the substring 10^t . Observe that \mathbf{S} satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations do not create the substring 10^t . Thus, \mathbf{S} is a flip-swap language.

Binary strings with forbidden prefix 1γ

Let \mathbf{S} be the set of binary strings of length n without the prefix 1γ . Observe that \mathbf{S} satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations either create a string with the prefix 0 or produce the same string. Thus, \mathbf{S} is a flip-swap language.

Lyndon words

Let $\mathbf{L}(n)$ denote the set of Lyndon words of length n . Since $\mathbf{N}(n)$ is a flip-swap language and $\mathbf{L}(n) \cup \{0^n\} \subseteq \mathbf{N}(n)$, it suffices to show that applying the flip-first or the swap-first operation on a Lyndon word either yields an aperiodic string or the string 0^n . Clearly $\mathbf{L}(n) \cup \{0^n\}$ satisfies the two closure properties of a flip-swap language when $\alpha \in \{0^n, 0^{n-1}1\}$. Thus in the remaining of the proof, $\alpha \notin \{0^n, 0^{n-1}1\}$. We first prove by contradiction that $\mathbf{L}(n) \cup \{0^n\}$ satisfies the flip-first closure property. Let $\alpha = 0^j 1 b_{j+2} b_{j+3} \cdots b_n$ be a string in $\mathbf{L}(n) \cup \{0^n\}$. Suppose that $\mathbf{L}(n) \cup \{0^n\}$ does not satisfy the flip-first closure property and $\text{flip}_\alpha(\ell_\alpha)$ is periodic. Thus $\text{flip}_\alpha(\ell_\alpha) = (0^{j+1}\beta)^t$ for some string β and $t \geq 2$. Observe that $\alpha = 0^j 1 \beta (0^{j+1}\beta)^{t-1}$ which is clearly not

a Lyndon word, a contradiction. Therefore $\mathbf{L}(n) \cup \{0^n\}$ satisfies the flip-first closure property.

Then similarly we prove by contradiction that $\mathbf{L}(n) \cup \{0^n\}$ satisfies the swap-first property. If $b_{j+2} = 1$, then applying the swap-first operation on α produces the same Lyndon word. Thus in the remaining of the proof, $b_{j+2} = 0$. Suppose that $\mathbf{L}(n) \cup \{0^n\}$ does not satisfy the swap-first closure property such that $\alpha \in \mathbf{L}(n) \cup \{0^n\}$ but $\text{swap}_\alpha(\ell_\alpha, \ell_\alpha + 1)$ is periodic. Thus $\text{swap}_\alpha(\ell_\alpha, \ell_\alpha + 1) = (0^{j+1}1\beta)^t$ for some string β and $t \geq 2$. Thus α contains the prefix 0^j1 but also the substring $0^{j+1}1$ in its suffix which is clearly not a Lyndon word, a contradiction. Thus, $\mathbf{L}(n)$ is a flip-swap language.

In [23], Vajnovszki proved that the BRGC order induces a cyclic 2-Gray code for the set of Lyndon words of length n .

Prenecklaces

Recall that a string α is a *prenecklace* if it is a prefix of some necklace. In Section 3 we prove that applying the flip-first or the swap-first operation on a necklace yields a necklace. Thus by the definition of prenecklace, applying the flip-first or the swap-first operation on a prenecklace also creates a string that is a prefix of a necklace. Thus, the set of prenecklaces of length n is a flip-swap language.

Pseudo-necklaces

Recall that a *block* with respect to 0^*1^* is a maximal substring of the form 0^*1^* . Each block B_i with respect to 0^*1^* can be represented by two integers (s_i, t_i) corresponding to the number of 0s and 1s respectively. For example, the string $\alpha = 000110100011001$ can be represented by $B_4B_3B_2B_1 = (3, 2)(1, 1)(3, 2)(2, 1)$. A block $B_i = (s_i, t_i)$ is said to be *lexicographically smaller* than a block $B_j = (s_j, t_j)$ (denoted by $B_i < B_j$) if $s_i < s_j$ or $s_i = s_j$ with $t_i < t_j$.

A string $\alpha = b_1b_2 \cdots b_n = B_bB_{b-1} \cdots B_1$ is a *pseudo-necklace* with respect to 0^*1^* if $B_b \leq B_i$ for all $1 \leq i < b$. Observe that the set of pseudo-necklaces of length n satisfies the two closure properties of a flip-swap language as the flip-first and swap-first operations do not make the first block B_b lexicographically larger, while the remaining blocks either remain the same or become lexicographically larger. Thus, the set of pseudo-necklaces of length n is a flip-swap language.

In [21], the authors proved that the BRGC order induces a cyclic 2-Gray code for the set of pseudo-necklaces of length n .

Left factors of k -ary Dyck words

Recall that a *k -ary Dyck word* is a binary string of length $n = tk$ with t copies of 1 and $t(k-1)$ copies of 0 such that every prefix has at most $k-1$ copies of 0 for every 1. It is well-known that k -ary Dyck words are in one-to-one correspondence with k -ary trees with t internal nodes. When $k = 2$, Dyck words are counted by the Catalan numbers

and are equivalent to *balanced parentheses*. As an example, 110100 is a 2-ary Dyck word and is also a balanced parentheses string while 100110 is not a 2-ary Dyck word nor a balanced parentheses because its prefix of length three contains more 0s than 1s. k -ary Dyck words and balanced parentheses strings are well studied and have lots of applications including trees and stack-sortable permutations [4, 16, 18, 24].

The set of k -ary Dyck words of length n is not a flip-swap language with respect to 0 since 110100 is a 2-ary Dyck word but 111100 is not. The set of length n prefixes of k -ary Dyck words is, however, a flip-swap language with respect to 0. This set is also called *left factors of k -ary Dyck words*. Let \mathbf{S} be the set of left factors of k -ary Dyck words. Observe that \mathbf{S} satisfies the two closure properties of a flip-swap language with respect to 0 as the flip-first and swap-first operations do not increase the number 0s in the prefix. Thus, \mathbf{S} is a flip-swap language with respect to 0.