University of Guelph, School of Computer Science, Prof. Pascal Matsakis

A Review of Functions

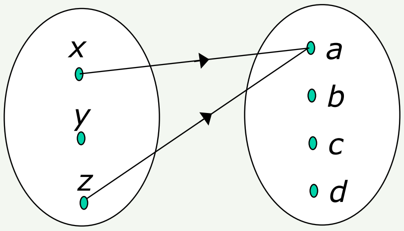
Domain, Codomain, Graph

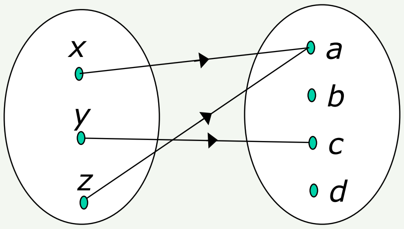
Let U and V be two sets. A ***function*** from U to V is a triple (U,V,G), where G is a subset of UxV, such that:

$$\forall u \in U,\ \forall (v_1, v_2) \in V^2,\ [(u, v_1) \in G \wedge (u, v_2) \in G] \rightarrow v_1 = v_2$$

U is the ***domain*** of the function, V is the ***codomain***, G is the ***graph***.

Domain is {x,y,z},
codomain is {a,b,c,d},
graph is {(x,a),(z,a)}.





Domain is {x,y,z},
codomain is {a,b,c,d},
graph is {(x,a),(y,c),(z,a)}.

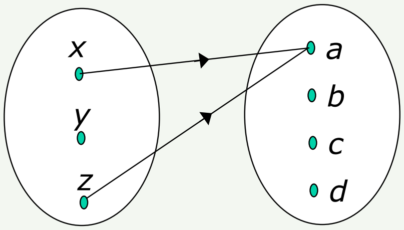
Domain of Definition, Range

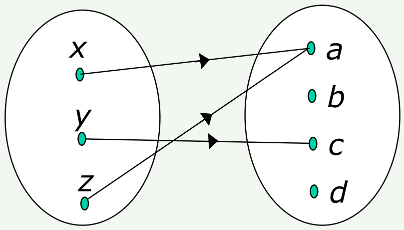
Consider a function f=(U,V,G).

$\{u \in U \mid \exists v \in V, (u,v) \in G\}$ is the ***domain of definition*** of f.
$\{v \in V \mid \exists u \in U, (u,v) \in G\}$ is the ***range*** of f.

If the domain of definition is U then f is a ***total function***.
If u belongs to the domain of definition then f is ***defined at*** u.
If U’ is a subset of the domain of definition then f is ***defined on*** U’.

Domain of definition is {x,z},
range is {a}.





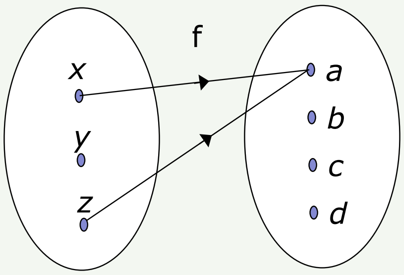
Domain of definition is {x,y,z},
range is {a,c}.

Images and Preimages of Elements

Consider a function f=(U,V,G).

Let (u,v) be an element of G.
v is the ***image*** of u under f; it is denoted by **f(u)** (read “f of u”).
u is a ***preimage*** of v under f.

$f(x)=a$, $f(\cancel{y})$, $f(z)=a$



Images and Preimages of Sets

Consider a function f=(U,V,G).

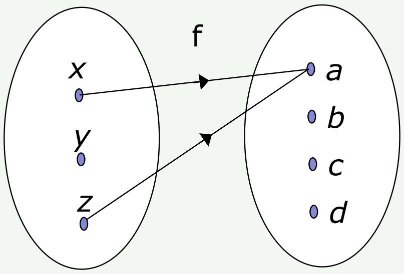
Let U' be a subset of U and let V' be a subset of V.

$\{v \in V \mid \exists u \in U', (u,v) \in G\}$ is the ***image*** of U' (under f);
it is denoted by **f(U′)**.

$\{u \in U \mid \exists v \in V', (u,v) \in G\}$ is the ***preimage*** of V' (under f);
it is denoted by **$f^{-1}(V')$** (read "f inverse of V′").

$f(\{x\})=f(\{z\})=\{a\}$, $f(\{y\})=\{\}$,
$f(\{x,y\})=f(\{x,y,z\})=\{a\}$

$f^{-1}(\{a\})=f^{-1}(\{a,b,c,d\})=\{x,z\}$,
$f^{-1}(\{b\})=f^{-1}(\{b,c,d\})=\{\}$



Examples and Notation

Consider the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2-\sqrt{x}$ (read "f is a function from $\mathbb{R}$ to $\mathbb{R}$ which maps x to $2-\sqrt{x}$")

Domain is $\mathbb{R}$ and codomain is $\mathbb{R}$.
Domain of definition is $[0,+\infty[$ and range is $]-\infty,2]$.
$\forall x \in [0,+\infty[$, $f(x)=2-\sqrt{x}$ (e.g., $f(\cancel{-}1)$, $f(0)=2$, $f(3)=2-\sqrt{3}$, $f(9)=-1$)

Consider the function $g : [-10,10] \to [0,+\infty[$, $x \mapsto 2-\sqrt{x}$

Domain is $[-10,10]$ and codomain is $[0,+\infty[$.
Domain of definition is $[0,4]$ and range is $[0,2]$.
$\forall x \in [0,4]$, $g(x)=2-\sqrt{x}$ (e.g., $g(\cancel{-}1)$, $g(0)=2$, $g(3)=2-\sqrt{3}$, $\cancel{g(}9)$)

More Notations

Consider a function f

Consider a function $f : x \mapsto f(x)$

Consider a function $x \mapsto f(x)$

Consider a function $f(x)$ **not allowed here**

same

Consider the* function $x \mapsto 2-\sqrt{x}$

Consider the* function $2-\sqrt{x}$ **not allowed here**

same

Consider the* function $g : x \mapsto 2-\sqrt{x}$

Consider a function $h \mid \mathbb{R} \to [0,+\infty[$

* Domain is $\mathbb{R}$? Codomain is $\mathbb{R}$?

Injections, Surjections, Bijections

Consider a function f from U to V.
Let D be the domain of definition of f.

f is ***injective***, i.e., it is an ***injection***, iff:
$\forall (u_1, u_2) \in D^2,\ f(u_1) = f(u_2) \to u_1 = u_2$

f is ***surjective***, i.e., it is a ***surjection***, iff:
$\forall v \in V,\ \exists u \in D,\ f(u) = v$

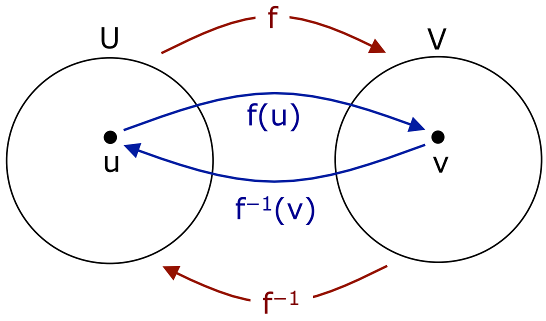
f is ***bijective***, i.e., it is a ***bijection***, iff:
f is an injective and surjective total function.

$f : \mathbb{R} \to \mathbb{R}$	$g : [0,+\infty[\to \mathbb{R}$	$h : \mathbb{R} \to \mathbb{R}^-$	$k : \mathbb{R}^+ \to \mathbb{R}^-$
$x \mapsto -1/x^2$	$x \mapsto -1/x^2$	$x \mapsto -1/x^2$	$x \mapsto -1/x^2$
	injective	surjective	bijective

Inverse

Consider a function $f=(U,V,G)$.

Let $G^{-1}=\{(v,u)\in V\text{x}U \mid (u,v)\in G\}$ (read “G inverse”) and $f^{-1}=(V,U,G^{-1})$.
If f^{-1} is a function, it is called the ***inverse function*** of f.
We then say that f is ***invertible***.

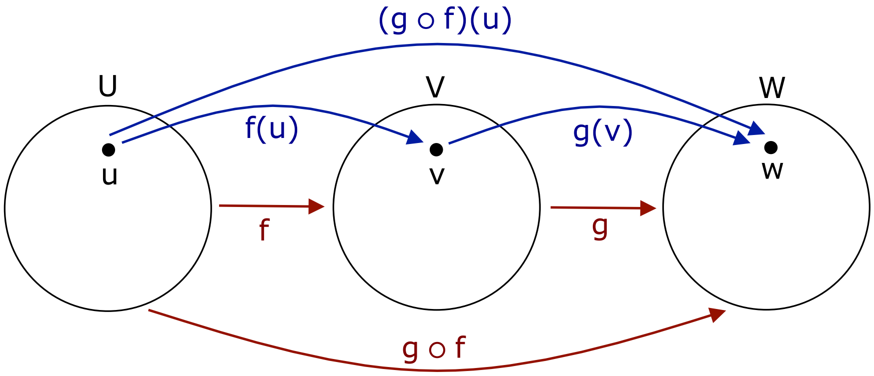


$f : \mathbb{R}\to\mathbb{R}$ $x \mapsto x^2$	f^{-1} is not a function	$g : \mathbb{R}\to\mathbb{R}$ $x \mapsto 1/(2x+1)$	$g^{-1} : \mathbb{R}\to\mathbb{R}$ $x \mapsto (1-x)/(2x)$

Composition

Consider two functions $f=(U,V,G)$ and $g=(V,W,H)$.
Let $K=\{(u,w)\in U\text{x}W \mid \exists v\in V, (u,v)\in G \wedge (v,w)\in H\}$.

The function (U,W,K) is the ***composition*** of f and g.
It is denoted by $g \circ f$ (read “g compose f).



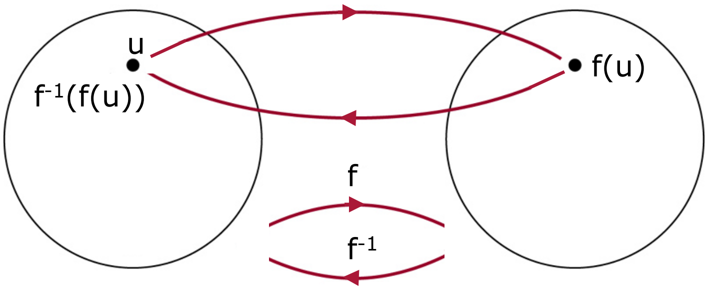
$f : \mathbb{R}\to\mathbb{R}$ $x \mapsto x^2$	$g : \mathbb{R}\to\mathbb{R}$ $x \mapsto 2x+1$	$g \circ f : \mathbb{R}\to\mathbb{R}$ $x \mapsto 2x^2+1$	$f \circ g : \mathbb{R}\to\mathbb{R}$ $x \mapsto (2x+1)^2$

Composition with the Inverse

Consider a bijective function f : U → U.
Then f^{-1} is a bijective function too,

$$\begin{array}{l} f \circ f^{-1} : U \to U \\ \qquad u \mapsto u \end{array} \quad \text{and} \quad \begin{array}{l} f^{-1} \circ f : U \to U \\ \qquad u \mapsto u \end{array}$$

$$\begin{array}{l} f : \mathbb{R} \to \mathbb{R} \\ \quad x \mapsto 3x - 1 \end{array} \qquad \begin{array}{l} f^{-1} : \mathbb{R} \to \mathbb{R} \\ \quad x \mapsto (x+1)/3 \end{array}$$



Unary and Binary Operations

A ***unary operation*** ~ on a set S is a function from S to S.
For any s in S, we usually write ~s instead of ~(s).

− (opposite) is a unary operation on $\mathbb{R}$.

A ***binary operation*** ★ on a set S is a function from S^2 to S.
For any s and t in S, we usually write s★t instead of ★((s,t)).

+ and × (addition and multiplication) are binary operations on $\mathbb{R}$.

Operations on Sets of Functions

Consider two functions f and g from U to V.
Let ~ be a unary operation on V and let ★ be a binary operation on V.

$$\mathbf{\sim f} : U \to V \qquad u \mapsto \sim(f(u))$$

$$\mathbf{f \star g} : U \to V \qquad u \mapsto f(u) \star g(u)$$

Consider two **numeric functions** f and g from U to V (and $V \subseteq \mathbb{R}$).

$$\sqrt{\mathbf{f}} : U \to V \qquad u \mapsto \sqrt{f(u)}$$

$$\mathbf{f^2} : U \to V \qquad u \mapsto [f(u)]^2$$

$$\frac{\mathbf{1}}{\mathbf{f}} : U \to V \qquad u \mapsto \frac{1}{f(u)}$$

$$|\mathbf{f}| : U \to V \qquad u \mapsto |f(u)|$$

$$\mathbf{f + g} : U \to V \qquad u \mapsto f(u) + g(u)$$

$$\mathbf{fg} : U \to V \qquad u \mapsto f(u)g(u)$$

$$\mathbf{f - g} : U \to V \qquad u \mapsto f(u) - g(u)$$

END