Pivot Gray Codes for the Spanning Trees of a Graph ft. the Fan

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Abstract

We consider the problem of listing all spanning trees of a graph G such that successive trees differ by pivoting a single edge around a vertex. Such a listing is called a "pivot Gray code", and it has more stringent conditions than known "revolving-door" Gray codes for spanning trees. Most revolving-door algorithms employ a standard edge-deletion/edge-contraction recursive approach which we demonstrate presents natural challenges when requiring the "pivot" property. Our main result is the discovery of a greedy strategy to list the spanning trees of the fan graph in a pivot Gray code order. It is the first greedy algorithm for exhaustively generating spanning trees using such a minimal change operation. The resulting listing is then studied to find a recursive algorithm that produces the same listing in $O(1)$-amortized time using $O(n)$ space. Additionally, we present $O(n)$-time algorithms for ranking and unranking the spanning trees for our listing. Finally, we discuss how our listing can be applied to find a pivot Gray code for the wheel graph.

1 Introduction

Applications of efficiently listing all spanning trees of general graphs are ubiquitous in computer science and also appear in many other scientific disciplines [6]. In fact, one of the earliest known works on listing all spanning trees of a graph is due to the German physicist Wilhelm Feussner in 1902 who was motivated by an application to electrical networks [10]. In the 120 years since Feussner's work, many new algorithms have been developed, such as those in the following citations [2, 3, 7, 9, 11, 12, 15, 16, 17, 21, 23, 24, 25, 27, 28, 29, 30, 31, 34].

For any application, it is desirable for spanning tree listing algorithms to have the asymptotically best possible running time, that is, $O(1)$-amortized running time. The algorithms due to Kapoor and Ramesh [17], Matsui [23], Smith [31], Shioura and Tamura [29] and Shioura et al. [30] all run in $O(1)$-amortized time. Another desirable property of such listings is to have the *revolving-door* property, where successive spanning trees differ by the addition of one edge and the removal of another. Such listings where successive objects in a listing differ by a constant number of simple operations are more generally known as *Gray codes*. The results on Gray codes for spanning trees of Kamae [16], Kishi and Kajitani [21], Holzmann and Harary [15], and Cummins [9] are all existential and do not give a method (efficient or not) of generating the Gray code. Only the algorithm due to Smith [31], and the recent algorithms due to Merino, Mütze, and Williams [27] and Merino and Mütze [25] produce Gray code listings of spanning trees for an arbitrary graph. Of these algorithms, Smith's is the only one that produces a Gray code listing in $O(1)$-amortized time.

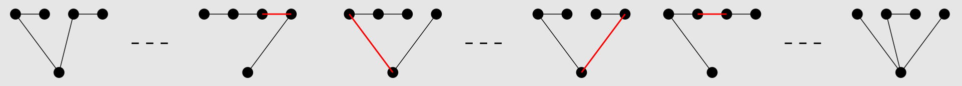
Example 1 Consider the fan graph on five vertices illustrated in Figure 1, where the seven edges are labeled

$$e_1 = v_2v_3,\ e_2 = v_2v_\infty,\ e_3 = v_3v_4,\ e_4 = v_3v_\infty,\ e_5 = v_4v_5,\ e_6 = v_4v_\infty,\ e_7 = v_5v_\infty.$$

The following is a revolving-door Gray code for the 21 spanning trees of this graph. The initial spanning tree has edges $\{e_1, e_2, e_5, e_6\}$ and each step of the listing below provides the edge that is removed from the current tree followed by the new edge that is added to obtain the next tree in the listing:

1. $\{e_1, e_2, e_5, e_6\}$
2. $-e_5 + e_7$
3. $-e_1 + e_3$
4. $-e_3 + e_4$
5. $-e_7 + e_5$
6. $-e_4 + e_3$
7. $-e_2 + e_1$
8. $-e_3 + e_4$
9. $-e_5 + e_7$
10. $-e_4 + e_3$
11. $-e_6 + e_5$
12. $\boldsymbol{-e_5 + e_2}$
13. $-e_2 + e_4$
14. $-e_3 + e_5$
15. $-e_4 + e_2$
16. $\boldsymbol{-e_7 + e_3}$
17. $-e_2 + e_4$
18. $-e_1 + e_2$
19. $-e_4 + e_7$
20. $-e_3 + e_4$
21. $-e_5 + e_3$.

This listing was generated from Knuth's implementation of Smith's [31] algorithm provided at `http://combos.org/span`. The steps in red are highlighted to show where the edge moves do not pivot around a vertex. This is further illustrated below:



1. 11. 12. 15. 16. 21.

A stronger notion of a Gray code for spanning trees is where the revolving-door makes strictly local changes. More specifically, we would like the edges being removed and added at each step to share a common endpoint. For applications that require an exhaustive listing of spanning trees, this is perhaps the minimal change we can hope for between successive spanning trees. We call a listing with this property a *pivot Gray code* (also known as a *strong revolving-door Gray code* [22]). The aforementioned spanning tree Gray codes are not pivot Gray codes. In particular, the Gray code given by Smith's algorithm [31] is not a pivot Gray code as illustrated in our previous example: the highlighted edge moves to obtain spanning trees 12 and 16 do not have the "pivot" property. This leads to our first research question.

Research Question #1 Given a graph G (perhaps from a specific class), does there exist a pivot Gray code listing of all spanning trees of G? Furthermore, can the listing be generated in polynomial time per tree using polynomial space?

A short discussion as to why previous methods do not lead directly to pivot Gray codes is presented in Section 1.3.

A related question that arises for any listing is how to *rank*, that is, find the position of the object in the listing, and *unrank*, that is, return the object at a specific rank. For spanning trees, Colbourn, Myrvold and Neufeld [8] give the best-known algorithm for ranking and unranking a spanning tree of a specific listing for an arbitrary graph; their algorithm has run-time equal to that of the fastest matrix multiplication algorithm, which is currently known to be $O(n^{2.371552})$-time [33].

Research Question #2 Given a graph G (perhaps from a specific class), does there exist a (pivot Gray code) listing of all spanning trees of G that can be ranked and unranked in $O(n^2)$ time?

An algorithmic technique recently found to have success in the discovery of Gray codes is the greedy approach. An algorithm is said to be *greedy* if it can prioritize allowable actions according to some criteria, and then choose the highest priority action that results in a unique object to obtain the next object in the listing. When applying a greedy algorithm, there is no backtracking; once none of the valid actions lead to a new object in the set under consideration, the algorithm halts, even if the listing is not exhaustive. The work by Williams [32] notes that some very well-known combinatorial listings can be constructed greedily, including the binary reflected Gray code (BRGC) for binary strings, the plain change order for permutations, and the lexicographically smallest de Bruijn sequence. Recently, a very powerful greedy algorithm on permutations (known as Algorithm J, where J stands for "jump") generalizes many known combinatorial Gray code listings including many related to permutation patterns, rectangulations, and elimination trees [13, 14, 26]. However, no greedy algorithm was previously known to list the spanning trees of an arbitrary graph.

Research Question #3 Given a graph G (perhaps from a specific class), does there exist a greedy strategy to list all spanning trees of G? Moreover, is the resulting listing a pivot Gray code?

In most cases, a greedy algorithm requires exponential space to recall which objects have already been visited in a listing. Thus, answering this third question would satisfy only the first part of **Research Question #1**. However, in many cases, an underlying pattern can be found in a greedy listing which can result in space efficient algorithms [13, 32]. In fact, the first part of this research question has been successfully addressed in recent work (following our initial submission) by Moreno, Mütze, and Williams [27]. They prove that the spanning trees of a graph, no matter the labeling of its edges, can be generated by the following greedy rule, for any initial spanning tree: Minimize the length of the prefix change to obtain a new tree. With additional restrictions, this approach yields a "face-pivot" Gray code for the fan; however, in general, this greedy approach does not yield a pivot Gray code. The *shortest prefix change* greedy approach has been extended to a wide variety of interesting combinatorial objects by Merino and Mütze [25].

To address these three research questions, we applied a variety of greedy approaches to structured classes of graphs including the fan, wheel, n-cube, and the complete graph. From this study, we were able to affirmatively answer each of the research questions for the fan graph. Furthermore, we adapt our pivot Gray code for the fan to obtain a pivot Gray code for the wheel. It remains an open question to find similar results for other classes of graphs, including the n-cube and the complete graph. Listing spanning trees for various plane graphs have also been studied [1, 19, 20], but they do not address the questions raised in this paper.

1.1 New results

The *fan graph* on n vertices, denoted F_n, is obtained by joining a single vertex (which we label v_∞) to the path on $n-1$ vertices (labeled $v_2, ..., v_n$) – see Figure 1. Note that we label the smallest vertex v_2 so that the largest non-infinity labeled vertex equals the total number of vertices. We discover a greedy strategy to generate the spanning trees of F_n in a pivot Gray code order. We describe this greedy

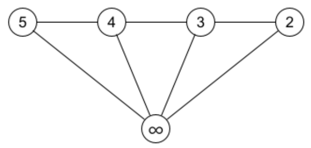


Figure 1 The fan F_5

strategy in Section 2. The resulting listing is studied to find an $O(1)$-amortized time recursive algorithm that produces the same listing using only $O(n)$ space, which is presented in Section 3. We show how to rank and unrank a spanning tree of our listing in $O(n)$ time in Section 3.2 and Section 3.3, which is a significant improvement over the general $O(n^3)$-time ranking and unranking that is already known. We conclude with a summary in Section 4, along with a discussion as to how our pivot Gray code for the fan can be extended to the wheel.

A complete C implementation of our algorithms is available in the Appendix. A preliminary version of this paper appeared in COCOON 2021 [5].

1.2 Oriented spanning trees

Although we only consider undirected graphs in this paper, we point out a related open problem for directed graphs.

Given a directed graph D and a fixed root vertex r, an *oriented spanning tree* or *spanning arborescence* is an oriented subtree T of D with $n-1$ arcs such that there is a unique path from r to every other vertex in D; all the arcs are directed away from r in T. The problem of finding a revolving-door Gray code for oriented spanning trees remains an open problem with a difficulty rating of 46/50 as given by Knuth in problem 102 on page 481 of [22]. Knuth also notes on page 804 that a solution to this problem for a fixed root r implies that a strong revolving-door (pivot) Gray code exists for the spanning trees of an undirected graph[1]. The mapping here is natural: given an undirected graph G, replace all edges (u, v) with two directed edges, one from u to v and one from v to u. Algorithms to list all oriented spanning trees with a given root are known [11, 18]; however, neither have the revolving-door property.

1.3 Edge contraction and deletion

A technique applied in the construction of several "revolving-door" Gray codes [28, 31, 34] is to recursively partition the spanning trees of a graph G into those containing a specific edge e by applying edge contraction, and those that do not contain the edge e by deleting e. However, when applying this strategy to construct a pivot Gray code, there are challenges when it comes to *uncontracting an edge*. Specifically, even if we have a pivot Gray code for the spanning trees in G/e (G with the edge e contracted), once we uncontract e, it does not necessarily result in a pivot Gray code for the original graph G. See, for example Figure 2. It is interesting to note, however, that if a different Gray code for G/e had been chosen in Figure 2 (i.e., swap the second and third trees), it would indeed result in a pivot Gray code for the spanning trees G that contain e. An interesting problem that remains open is if there are graphs for which there is no pivot Gray code that can be "lifted" from their deletion-contraction Gray codes.

2 A greedy approach

In this section, we discuss greedy approaches that can be applied to discover pivot Gray codes. The graphs we consider are the complete graph K_n, the fan F_n, the wheel W_n, and the n-cube. Note that every unicyclic graph, that is, a graph with exactly one cycle, has a

[1] The author's thank Torsten Mütze for pointing out this comment.

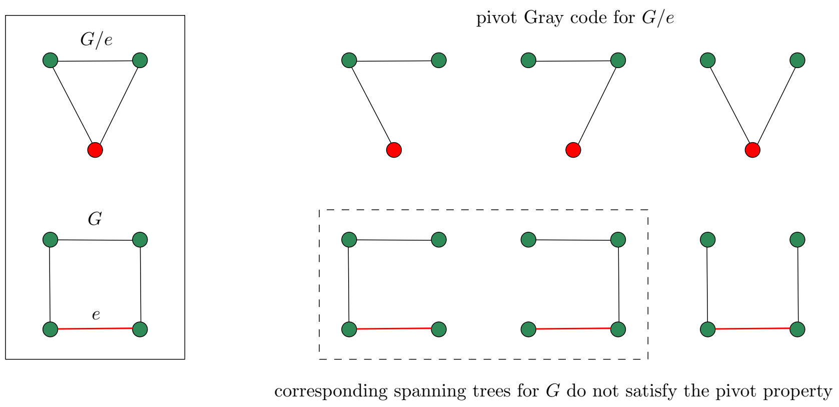

pivot Gray code for G/e
G/e
G
e
corresponding spanning trees for G do not satisfy the pivot property

Figure 2 Illustrating how a pivot Gray code in graph G/e does not necessarily correspond to a pivot Gray code for the spanning trees of G that include e.

cyclic pivot Gray code: start with any initial spanning tree then pivot the edges around the cycle one at time. There are two important issues when considering a greedy approach to list spanning trees: (1) the labels on the vertices (or edges) and (2) the starting tree. For each of our approaches, we prioritized our operations by first considering which vertex u to pivot on, followed by an ordering of the endpoints considered in the addition/removal. We call the vertex u the *pivot*.

Our initial attempts focused only on pivots that were leaves of the current spanning tree. As a specific example, we ordered the leaves (pivots) from smallest to largest. Since each leaf u is attached to a unique vertex v in the current spanning tree, we then considered the neighbours of u in increasing order of label. We restricted the labeling of the vertices to the most natural ones, such as the one presented in Section 1.1 for the fan. For each strategy we tried all possible starting trees. For each starting tree, we iterate the greedy rule until no new tree can be found via a pivot of the last tree; if all trees have been visited, then a pivot Gray code has been discovered. Unfortunately, none of our attempts lead to exhaustive listings for K_n, F_n, W_n, or the n-cube.

By allowing the pivot to be any arbitrary vertex, we ultimately discovered several exhaustive listing for the spanning trees of F_n; however, rather interestingly, we found no such listing for any other class. The listings we found for the fan were generated up to $n = 12$. Starting from every starting tree for $n = 12$ took about 8 hours on a single processor. One listing stood out as having an easily defined starting tree as well as a nice pattern which we could study to construct the listing more efficiently. It applied the labeling of the vertices as described in Section 1.1 with the following prioritization of pivots and their incident edges:

> Prioritize the pivots u from smallest to largest and then for each pivot, prioritize the edges uv that can be removed from the current tree in increasing order of the label on v, and for each such v, prioritize the edges uw that can be added to the current tree in increasing order of the label on w.

Since this is a greedy strategy, if an edge pivot results in a spanning tree that has already been generated or a graph that is not a spanning tree, then the next highest priority edge pivot is attempted. Let GREEDY(T) denote the listing that results from applying this greedy approach starting with the spanning tree T. The starting tree that produced a nice exhaustive listing was the path $v_\infty, v_2, v_3, \ldots, v_n$, denoted P_n throughout the paper. Figure 3 shows the listings GREEDY(P_n) for $n = 2, 3, 4, 5$. The listing GREEDY(P_6) is illustrated in Figure 4. It is worth noting that starting with the path $v_\infty, v_n, v_{n-1}, \ldots, v_2$ or the star (all edges incident to v_∞) did not lead to an exhaustive listing for the spanning trees of F_n in our study.

Example 2 Consider the listing $\textsc{Greedy}(P_5)$ in Figure 3. When the current tree T is the 16th one in the listing (the one with edges $\{v_2v_\infty, v_2v_3, v_3v_4, v_5v_\infty\}$), the first pivot considered is v_2. Since both v_2v_3 and v_2v_∞ are present in the tree, no valid move is available by pivoting on v_2. The next pivot considered is v_3. Both edges v_3v_2 and v_3v_4 are incident with v_3. First, we attempt to remove v_3v_2 and add v_3v_∞, which results in a tree previously generated. Next, we attempt to remove v_3v_4 and add v_3v_∞, which results in a cycle. So, the next pivot, v_4, is considered. The only edge incident to v_4 is v_4v_3. By removing v_4v_3 and adding v_4v_5 we obtain a new spanning tree, the next tree in the greedy listing.

To prove that $\textsc{Greedy}(P_n)$ does in fact contain all the spanning trees of F_n, the next section demonstrates it is equivalent to a recursively constructed listing obtained by studying the greedy listings. Before we describe this recursive construction we mention one rather remarkable property of $\textsc{Greedy}(P_n)$ that we prove later in Section 3.1.1.

▶ Remark 1. Let X_n be last tree in the listing $\textsc{Greedy}(P_n)$. Then $\textsc{Greedy}(X_n)$ is precisely $\textsc{Greedy}(P_n)$ in reverse order.

3 A pivot Gray code for the spanning trees of F_n via recursion

In this section we develop an efficient recursive algorithm to construct the listing $\textsc{Greedy}(P_n)$. The construction generates some sub-lists in reverse order, similar to the recursive construction of the BRGC. The recursive properties allow us to provide efficient ranking and unranking algorithms for the listing based on counting the number of trees at each stage of the construction. Let t_n denote the number of spanning trees of F_n. It is known that

$$t_n = f_{2(n-1)} = 2\frac{((3-\sqrt{5})/2)^n - ((3+\sqrt{5})/2)^{n-2}}{5-3\sqrt{5}},$$

where f_n is the nth number of the Fibonacci sequence with $f_1 = f_2 = 1$ [4].

3.1 Pivot Gray code construction

By studying the order of the spanning trees in $\textsc{Greedy}(P_n)$, we identified four distinct stages S1, S2, S3, S4 that are highlighted for $\textsc{Greedy}(P_6)$ in Figure 4. From this figure, and referring back to Figure 3 to see the recursive properties, observe that:

- The trees in S1 are equivalent to $\textsc{Greedy}(P_5)$ with the added edge v_6v_5.
- The trees in S2 are equivalent to the reversal of the trees in $\textsc{Greedy}(P_5)$ with the added edge v_6v_∞.

The trees in S3 and S4 have both edges v_6v_5 and v_6v_∞ present.

- In S3, focusing only on the vertices v_4, v_3, v_2, v_∞, the induced subgraphs correspond to $\textsc{Greedy}(P_4)$, except whenever v_4v_∞ is present, it is replaced with v_4v_5 (the last five trees).
- In S4, focusing only on the vertices v_4, v_3, v_2, v_∞, the induced subgraphs correspond to the trees in $\textsc{Greedy}(P_4)$ where v_4v_∞ is present, in reverse order.

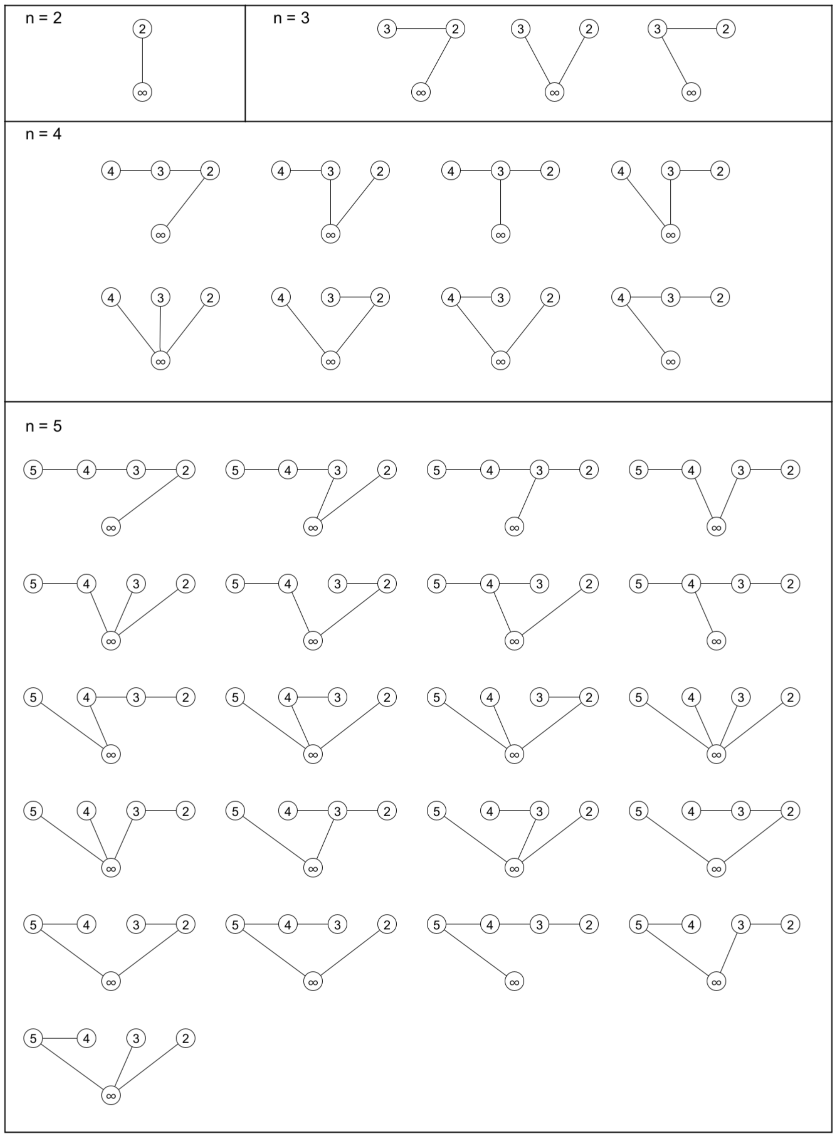


Figure 3 Greedy(P_n) for $n = 2, 3, 4, 5$. Read left to right, top to bottom.

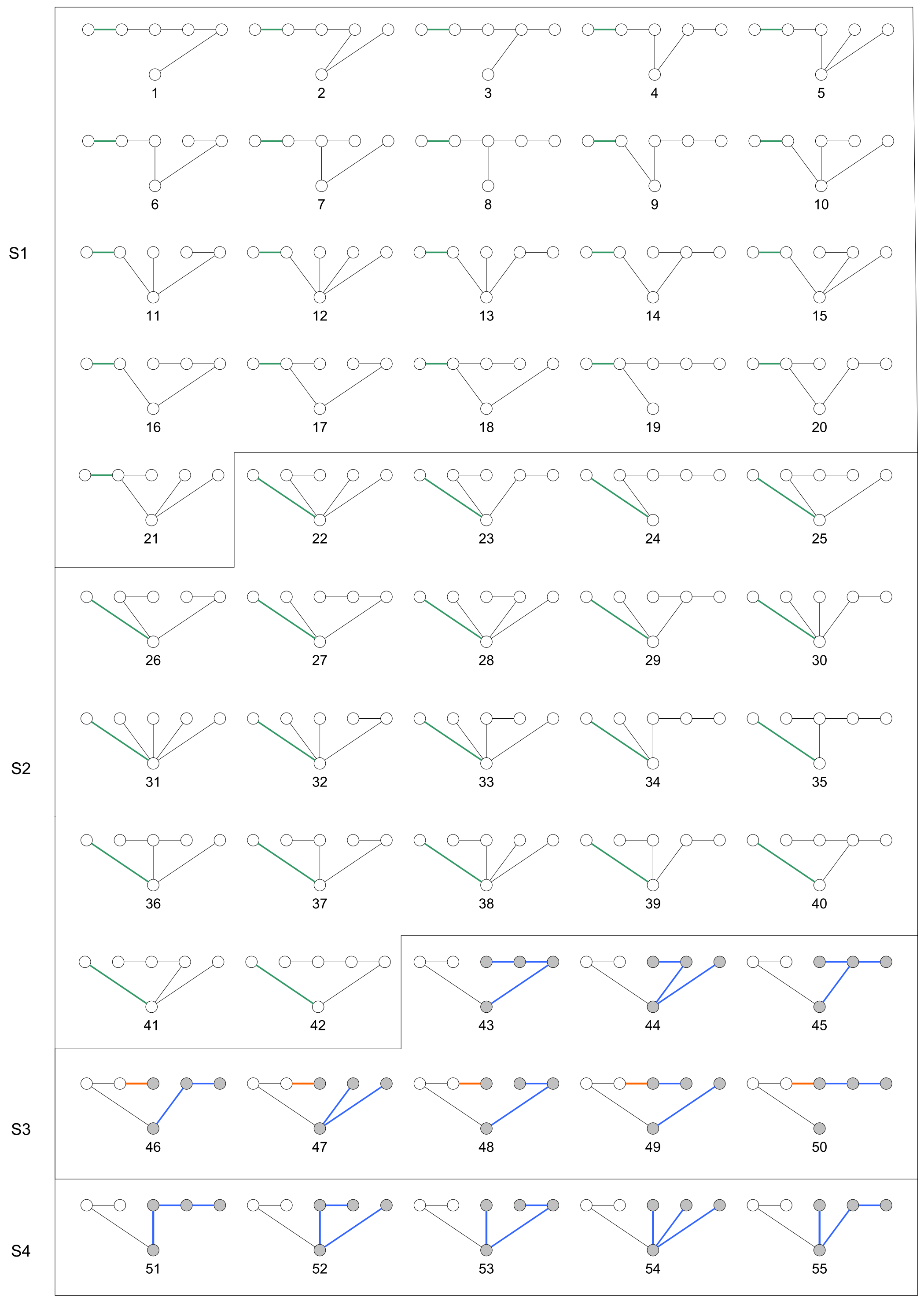


Figure 4 GREEDY(P_6) read from left to right, top to bottom. Observe that S1 is GREEDY(P_5) with v_6v_5 added, S2 is the reverse of GREEDY(P_5) with v_6v_∞ added, S3 is GREEDY(P_4) with v_6v_5 and v_6v_∞ added, except the edge v_4v_∞ is replaced by v_4v_5, and S4 is the last five trees of GREEDY(P_4) in reverse order (v_4v_∞ is now present) with v_6v_5 and v_6v_∞ added.

Algorithm 1 Generate the spanning trees of the fan F_n in pivot Gray code order. T is initialized to P_n and printed before the call $\textsc{Gen}(n, 1, 0)$.

```
1:  procedure Gen($k, s_1, varEdge$)
2:      if $k = 2$ then                                              ▷ $F_2$ base case
3:          if $varEdge$ then $T \leftarrow T - v_2v_\infty + v_2v_3$; Print($T$)
4:      else if $k = 3$ then                                         ▷ $F_3$ base case
5:          if $s_1$ then
6:              if $varEdge$ then $T \leftarrow T - v_3v_2 + v_3v_4$; Print($T$)
7:              else $T \leftarrow T - v_3v_2 + v_3v_\infty$; Print($T$)
8:          $T \leftarrow T - v_2v_\infty + v_2v_3$; Print($T$)
9:      else
10:         if $s_1$ then
11:             Gen($k-1, 1, 0$)                                     ▷ S1
12:             if $varEdge$ then $T \leftarrow T - v_kv_{k-1} + v_kv_{k+1}$; Print($T$)
13:             else $T \leftarrow T - v_kv_{k-1} + v_kv_\infty$; Print($T$)
14:         RevGen($k-1, 1, 0$)                                      ▷ S2
15:         $T \leftarrow T - v_{k-1}v_{k-2} + v_{k-1}v_k$; Print($T$)
16:         Gen($k-2, 1, 1$)                                         ▷ S3
17:         if $k > 4$ then $T \leftarrow T - v_{k-2}v_{k-1} + v_{k-2}v_\infty$; Print($T$)
18:         RevGen($k-2, 0, 0$)                                      ▷ S4
```

Generalizing these observations for all $n \geq 2$ leads to the recursive procedure given in Algorithm 1, called $\textsc{Gen}(k, s_1, \mathit{varEdge})$. It uses a global variable T to store the current spanning tree with n vertices. The parameter k indicates the number of vertices under consideration; the parameter s_1 indicates whether or not to generate the trees in stage S1, as required by the trees for S4; and the parameter *varEdge* indicates whether or not a variable edge needs to be added as required by the trees for S3. The base cases correspond to the edge moves in the listings $\textsc{Greedy}(P_2)$ and $\textsc{Greedy}(P_3)$.

Let $\mathcal{G}_n$ denote the listing obtained by initializing T to P_n, printing T, and calling $\textsc{Gen}(n, 1, 0)$.

Before discussing $\textsc{RevGen}$, we first provide a formal description of the last tree in the listing $\mathcal{G}_n$, which we denote L_n. Define the tree $Last_n$ as follows for $n \geq 2$: for $n = 2, 3, 4$ let $Last_n$ be the last trees in the listings for $n = 2, 3, 4$ given in Figure 3, and for $n > 4$ let

$$Last_n = Last_{n-3} + v_nv_{n-1} + v_nv_\infty + v_{n-2}v_\infty.$$

Applying this definition, the trees $Last_n$ for $2 \leq n \leq 7$ are given in Figure 5.

▶ **Lemma 2.** *For $n \geq 2$, $L_n = Last_n$.*

Proof. The proof is by induction on n, tracing the routines $\textsc{Gen}$ and $\textsc{RevGen}$. By definition of $Last_n$, the result holds for $n = 2, 3, 4$. Assume that $L_j = Last_j$ for $2 \leq j < n$. Recall that L_n is the last tree generated by a call to $\textsc{Gen}(n, 1, 0)$ when starting with the tree P_n. Tracing this routine following the proof of Lemma 6, the current spanning tree when calling $\textsc{RevGen}(n-2, 0, 0)$ (on line 18) is $L_{n-2} + v_nv_{n-1} + v_nv_\infty$. Thus from the definition of $Last_n$, we must show that the last tree of $\textsc{RevGen}(n-2, 0, 0)$ when starting with L_{n-2} is $Last_{n-3} + v_{n-2}v_\infty$.

Since $\textsc{RevGen}(k, 1, 0)$ starting with L_k is the reversal of $\textsc{Gen}(k, 1, 0)$ starting with P_k, then the last tree of S2 of $\textsc{RevGen}(k, 1, 0)$ must be the first tree of S2 of $\textsc{Gen}(k, 1, 0)$. The last tree of the recursive call S1 of $\textsc{Gen}(k, 1, 0)$ when starting with P_k is $Last_{k-1} + v_kv_{k-1}$ because, by the inductive hypothesis, $L_{k-1} = Last_{k-1}$, and $v_kv_{k-1} \in P_k$. Then, the edge

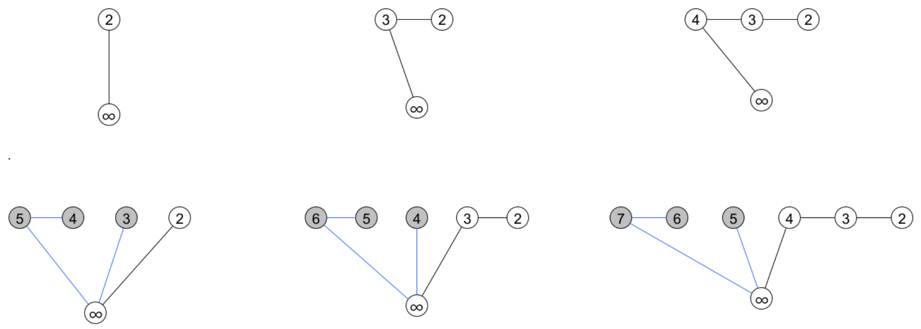


Figure 5 From left to right. $\{Last_2, Last_3, Last_4\}$ (top), $\{Last_5, Last_6, Last_7\}$ (bottom). Shaded vertices and blue edges highlight the additional vertices and edges added to the trees in the top row to obtain the trees in the bottom row.

Algorithm 2 Generate the spanning trees of the fan F_n in pivot Gray code order. T is initialized to L_n and printed before the call RevGen$(n, 1, 0)$.

```
1: procedure RevGen($k, s_1, varEdge$)
2:     if $k = 2$ then                                              ▷ $F_2$ base case
3:         if $varEdge$ then $T \leftarrow T - v_2v_3 + v_2v_\infty$; Print($T$)
4:     else if $k = 3$ then                                         ▷ $F_3$ base case
5:         $T \leftarrow T - v_2v_3 + v_2v_\infty$; Print($T$)
6:         if $s_1$ then
7:             if $varEdge$ then $T \leftarrow T - v_3v_4 + v_3v_2$; Print($T$)
8:             else $T \leftarrow T - v_3v_\infty + v_3v_2$; Print($T$)
9:     else
10:        Gen($k-2, 0, 0$)                                          ▷ S4
11:        if $k > 4$ then $T \leftarrow T - v_{k-2}v_\infty + v_{k-2}v_{k-1}$; Print($T$)
12:        RevGen($k-2, 1, 1$)                                       ▷ S3
13:        $T \leftarrow T - v_{k-1}v_k + v_{k-1}v_{k-2}$; Print($T$)
14:        Gen($k-1, 1, 0$)                                          ▷ S2
15:        if $s_1$ then
16:            if $varEdge$ then $T \leftarrow T - v_kv_{k+1} + v_kv_{k-1}$; Print($T$)
17:            else $T \leftarrow T - v_kv_\infty + v_kv_{k-1}$; Print($T$)
18:            RevGen($k-1, 1, 0$)                                   ▷ S1
```

move made by line 13 of Gen$(k, 1, 0)$ removes v_kv_{k-1} and adds v_kv_∞. It follows that the first tree of S2 of Gen$(k, 1, 0)$, and equivalently the last tree of S2 of RevGen$(k, 1, 0)$, which is also the last tree of RevGen$(k, 0, 0)$ when starting with L_k, is $Last_{k-1} + v_kv_\infty$. Thus, the last tree of RevGen$(n-2, 0, 0)$ starting with L_{n-2} is $Last_{n-3} + v_{n-2}v_\infty$, as desired. ◀

The procedure RevGen$(k, s_1, varEdge)$, performs the operations of Gen$(k, s_1, varEdge)$ in reverse order, thus producing the reversal of the listing generated by Gen$(k, s_1, varEdge)$ when starting with the last tree from the latter listing. The base cases correspond to the edge moves in the reversals of the listings Greedy(P_2) and Greedy(P_3).

Let $\mathcal{R}_n$ denote the listing obtained by initializing T to L_n, printing T, and calling RevGen$(n, 1, 0)$.

▶ Remark 3. $\mathcal{R}_n$ is the listing $\mathcal{G}_n$ in reverse order.

▶ **Theorem 4.** *For $n \geq 2$, $\mathcal{G}_n$ and $\mathcal{R}_n$ are pivot Gray codes for the spanning trees of the fan F_n and they can be generated in $O(1)$-amortized time using $O(n)$ space. Moreover, $\text{Greedy}(P_n) = \mathcal{G}_n$ and $\text{Greedy}(L_n) = \mathcal{R}_n$.*

3.1.1 Proof of Theorem 4

We start by proving that the number of trees generated by $\mathcal{G}_n$ is t_n. Then we show that $\mathcal{G}_n = \text{Greedy}(P_n)$ and $\mathcal{R}_n = \text{Greedy}(L_n)$. Combining these results with the fact that the trees generated by the greedy approaches are unique and successive trees differ by the "pivot" of a single edge, we have that $\mathcal{G}_n$ $\mathcal{R}_n$ are pivot Gray codes for the spanning trees of the fan graph F_n. Finally, we verify the running time of the recursive algorithm to generate $\mathcal{G}_n$.

Before proving these results, we introduce some notation. Let $T - v_i$ denote the tree obtained from T by deleting the vertex v_i along with all edges that have v_i as an endpoint. Let $T + v_iv_j$ (resp. $T - v_iv_j$) denote the tree obtained from T by adding (resp. deleting) the edge v_iv_j. For the remainder of this section, we will let T_n denote the tree T specified as a global variable for Gen and RevGen, and we let $T_{n-1} = T - v_n$ and $T_{n-2} = T - v_n - v_{n-1}$.

▶ **Lemma 5.** *For $n \geq 2$, $|\mathcal{G}_n| = |\mathcal{R}_n| = t_n$.*

Proof. We first note that since Gen and RevGen are exact reversals of each other, $\text{Gen}(n, s_1, \mathit{varEdge})$ starting with $T = P_n$ and $\text{RevGen}(n, s_1, \mathit{varEdge})$ starting with $T = L_n$ produce the same number of trees. The proof now proceeds by induction on n. It is easy to verify the result holds for $n = 2, 3, 4$. Now assume $n > 4$, and that $|\mathcal{G}_j| = t_j$, for $2 \leq j < n$. We consider the number of trees generated by each of the four stages of $\text{Gen}(n-1, 1, 0)$ when starting with P_n.

S1: Since $n > 4$ and $s_1 = 1$, $\text{Gen}(n-1, 1, 0)$ is executed. Since $T_n = P_n$, we have that $T_{n-1} = P_{n-1}$. So, by our inductive hypothesis, t_{n-1} trees are printed. By definition of L_n, $T_{n-1} = L_{n-1}$ after $\text{Gen}(n-1, 1, 0)$. It follows that $T_n = L_{n-1} + v_nv_{n-1}$. Line 12 removes v_nv_{n-1} and adds v_nv_∞. Since $v_nv_{n-1} \in T$ and $v_nv_\infty \notin T$, this results in one more tree printed. At this point, $T_n = L_{n-1} + v_nv_\infty$.

S2: Next, line 14 executes $\text{RevGen}(n-1, 1, 0)$. We have that $T_{n-1} = L_{n-1}$. So, by the inductive hypothesis, t_{n-1} trees are printed. We know that $T_{n-1} = P_{n-1}$ after $\text{RevGen}(n-1, 1, 0)$ starting with $T_{n-1} = L_{n-1}$, so it follows that $T_n = P_{n-1} + v_nv_\infty$. Line 15 removes $v_{n-2}v_{n-1}$ and adds $v_{n-1}v_n$. Since $v_{n-2}v_{n-1} \in T$ (because $v_{n-2}v_{n-1} \in P_{n-1}$) and $v_{n-1}v_n \notin T$, this results in one more tree printed. At this point, $T_n = P_{n-2} + v_nv_\infty + v_{n-1}v_n$.

S3: Line 16 then executes $\text{Gen}(n-2, 1, 1)$ with $T_{n-2} = P_{n-2}$ since $T_n = P_{n-2} + v_nv_\infty + v_{n-1}v_n$. Note that the only difference between $\text{Gen}(n-2, 1, 1)$ and $\text{Gen}(n-2, 1, 0)$ is that v_jv_{j+1} is added instead of v_jv_∞ since $n-2 > 2$. Also, $v_{n-2}v_{(n-2)+1} \notin T_n$ so it can be added. It follows that $\text{Gen}(n-2, 1, 1)$ and $\text{Gen}(n-2, 1, 0)$ will output the same number of trees starting with $T_{n-2} = P_{n-2}$. So, line 16 results in t_{n-2} trees printed, again by the inductive hypothesis. After line 16 is executed, we have $T_{n-2} = L_{n-2} - v_{n-2}v_\infty + v_{n-2}v_{n-1}$ since $\mathit{varEdge}$ was equal to 1. Line 17 removes $v_{n-2}v_{n-1}$ and adds $v_{n-2}v_\infty$. Since $v_{n-2}v_{n-1} \in T$ and $v_{n-2}v_\infty \notin T$, this results in one more tree printed. At this point, $T_n = L_{n-2} + v_nv_{n-1} + v_nv_\infty$.

S4: By our inductive hypothesis, $|\mathcal{R}_{n-2}| = t_{n-2}$. However, $s_1 = 0$ for line 18 ($\text{RevGen}(n-2, 0, 0)$). So, for $\text{RevGen}(n-2, 0, 0)$, line 17 (one tree) and line 18 (t_{n-3} trees) are not executed. This results in a total of $t_{n-2} - t_{n-3}$ trees being printed by line 18 of $\text{Gen}(n, 1, 0)$.

In total, $2t_{n-1} + 2t_{n-2} - t_{n-3}$ trees are printed. By a straightforward Fibonacci identity which we leave to the reader, we have that $t_n = 2t_{n-1} + 2t_{n-2} - t_{n-3}$. Therefore, $|\mathcal{G}_n| = |\mathcal{R}_n| = t_n$. ◀

To prove the next result, we first detail some required terminology. If T is a spanning tree of F_n, then we say that the operation of deleting an edge v_iv_j and adding an edge v_iv_k is a *valid* edge move of T if the result is a spanning tree that has not been generated yet. Conversely, if the result is not a spanning tree, or the result is a tree that has already been generated, then it is not a *valid* edge move of T. We say an edge v_iv_j is *smaller* than edge v_iv_k if $j < k$. An edge move $T_n - v_iv_j + v_iv_k$ is said to be *smaller* than another edge move $T_n - v_xv_y + v_xv_z$ if $i < x$, if $i = x$ and $j < y$, or if $i = x$, $j = y$, and $k < z$.

▶ **Lemma 6.** *For $n \geq 2$, $\mathcal{G}_n = \text{GREEDY}(P_n)$ and $\mathcal{R}_n = \text{GREEDY}(L_n)$.*

Proof. By induction on n. It is straightforward to verify that the result holds for $n = 2, 3, 4$ by iterating through the algorithms. Assume $n > 4$, and that $\mathcal{G}_j = \text{GREEDY}(P_j)$ and $\mathcal{R}_j = \text{GREEDY}(L_j)$ for $2 \leq j < n$. We begin by showing $\mathcal{G}_n = \text{GREEDY}(P_n)$, breaking the proof into each of the four stages of a call to $\text{GEN}(n-1, 1, 0)$ starting with P_n.

S1: Since $n > 4$ and $s_1 = 1$, $\text{GEN}(n-1, 1, 0)$ is executed. By our inductive hypothesis, $\mathcal{G}_{n-1} = \text{GREEDY}(P_{n-1})$. These must be the first trees for $\text{GREEDY}(P_n)$, as any edge move involving v_nv_{n-1} or v_nv_∞ is larger than any edge move made by $\text{GREEDY}(P_{n-1})$. Since $\text{GREEDY}(P_{n-1})$ halts, it must be that no edge move of T_{n-1} is possible. So $\text{GREEDY}(P_n)$ must make the next smallest edge move, which is $T_n - v_nv_{n-1} + v_nv_\infty$. Since T_n is a spanning tree, it follows that $T_n - v_nv_{n-1} + v_nv_\infty$ is also a spanning tree (and has not been generated yet), and therefore the edge move is valid. At this point, $\text{GEN}(n, 1, 0)$ also makes this edge move, by line 13.

S2: $\text{REVGEN}(n-1, 1, 0)$ $(T_{n-1} = L_{n-1})$ is then executed. By our inductive hypothesis, $\mathcal{R}_n = \text{GREEDY}(L_{n-1})$. Since $\text{GREEDY}(L_{n-1})$ halts, it must be that no edge moves of T_{n-1} are possible. At this point, $T_{n-1} = P_{n-1}$ because $\text{REVGEN}(n-1, 1, 0)$ was executed. The smallest edge move now remaining is $T_n - v_{n-2}v_{n-1} + v_nv_{n-1}$. This results in $T_n = P_{n-2} + v_nv_{n-1} + v_nv_\infty$, which is a spanning tree that has not been generated. So, $\text{GREEDY}(P_n)$ must make this move. $\text{GEN}(n, 1, 0)$ also makes this move, by line 15. So, $\mathcal{G}_n$ must equal $\text{GREEDY}(P_n)$ up to the end of S2.

S3: Next, $\text{GEN}(n-2, 1, 1)$ starting with $T_{n-2} = P_{n-2}$ is executed. Since $\mathit{varEdge} = 1$, $v_{n-2}v_{n-1}$ is added instead of $v_{n-2}v_\infty$. $\text{GREEDY}(P_n)$ also adds $v_{n-2}v_{n-1}$ instead of $v_{n-2}v_\infty$ since $v_{n-2}v_{n-1}$ is smaller than $v_{n-2}v_\infty$ and this edge move results in a tree not yet generated. Other than the difference in this one edge move, which occurs outside the scope of T_{n-2}, $\text{GEN}(n-2, 1, 0)$ and $\text{GEN}(n-2, 1, 1)$ (both starting with $T_{n-2} = P_{n-2}$) make the same edge moves. Since we also know that $\mathcal{G}_{n-2} = \text{GREEDY}(P_{n-2})$ by the inductive hypothesis, it follows that $\mathcal{G}_n$ continues to equal $\text{GREEDY}(P_n)$ after line 16 of $\text{GEN}(n, 1, 0)$ is executed. We know that $T_{n-2} = L_{n-2}$ after $\text{GEN}(n-2, 1, 0)$. However, $T_{n-2} = L_{n-2} - v_{n-2}v_\infty + v_{n-2}v_{n-1}$ instead because $\text{GEN}(n-2, 1, 1)$ was executed ($\mathit{varEdge} = 1$). It must be that no edge moves of T_{n-2} are possible because $\text{GREEDY}(P_{n-2})$ (and $\text{GEN}(n-2, 1, 1)$) halted. The smallest edge move now remaining is $T_n - v_{n-2}v_{n-1} + v_{n-2}v_\infty$. This results in $T_{n-2} = L_{n-2}$. Also, $T_n = T_{n-2} + v_nv_{n-1} + v_nv_\infty$ is a spanning tree since T_{n-2} is a spanning tree of F_{n-2}. So $\text{GREEDY}(P_n)$ makes this move.

$\textsc{Gen}(n, 1, 0)$ also makes this move, by line 17, and thus $\mathcal{G}_n = \textsc{Greedy}(P_n)$ up to the end of S3.

S4: Finally, $\textsc{RevGen}(n-2, 0, 0)$ starting with $T_{n-2} = L_{n-2}$ is executed. By our inductive hypothesis, $\mathcal{R}_{n-2} = \textsc{Greedy}(L_{n-2})$. From lines 15-18 of Algorithm 2, it is clear that $\textsc{RevGen}(n-2, 0, 0)$ and $\textsc{RevGen}(n-2, 1, 0)$ make the same edge moves until $\textsc{RevGen}(n-2, 0, 0)$ finishes executing. So, by the inductive hypothesis, the listings produced by $\textsc{RevGen}(n-2, 0, 0)$ and $\textsc{Greedy}(L_{n-2})$ are the same until this point, which is where $\textsc{Gen}(n, 1, 0)$ finishes execution. By Lemma 5 we have that $|\mathcal{G}_n| = t_n$. Therefore, $\textsc{Greedy}(P_n)$ has also produced this many trees, and each tree is unique. Thus, it must be that all t_n trees of F_n have been generated. Thus, $\textsc{Greedy}(P_n)$ also halts.

Since $\mathcal{G}_n$ and $\textsc{Greedy}(P_n)$ start with the same tree, produce the same trees in the same order, and halt at the same place, it follows that $\mathcal{G}_n = \textsc{Greedy}(P_n)$. We now prove that $\mathcal{R}_n = \textsc{Greedy}(L_n)$. By using our inductive hypothesis and the same arguments as previously, we see that this results hold within the four stages of $\textsc{RevGen}(n, 1, 0)$ when starting with L_n(namely, $\textsc{Gen}(T_{n-2}, 0, 0)$, $\textsc{RevGen}(T_{n-2}, 1, 1)$, $\textsc{Gen}(T_{n-1}, 1, 0)$, and $\textsc{RevGen}(T_{n-1}, 1, 0)$). However, we must still prove that $\mathcal{R}_n$ matches $\textsc{Greedy}(L_n)$ as we move between stages.

After S4: After $\textsc{Gen}(n-2, 0, 0)$, $T_n = L_{n-2} + v_n v_{n-1} + v_n v_\infty$. Since $v_{n-2} v_\infty \in T_n$ and no edge moves of T_{n-2} are possible (because $\textsc{Greedy}(L_{n-2})$ halted), $\textsc{Greedy}$ makes the next smallest edge move which is $T_n - v_{n-2} v_\infty + v_{n-2} v_{n-1}$. $\textsc{RevGen}(n, 1, 0)$ also makes this move here, by line 11.

After S3: After $\textsc{RevGen}(n-2, 1, 1)$, $T_n = P_{n-2} + v_n v_{n-1} + v_n v_\infty$. No edge moves of T_{n-2} are possible at this point. Therefore, since $v_n v_{n-1} \in T_n$ and $v_{n-2} v_{n-1} \notin T_n$, $\textsc{Greedy}$ makes the smallest possible edge move which is $T_n - v_n v_{n-1} + v_{n-2} v_{n-1}$. $\textsc{RevGen}(n, 1, 0)$ also makes this move here, by line 13.

After S2: Finally, after $\textsc{Gen}(n-1, 1, 0)$, $T_n = L_{n-1} + v_n v_\infty$. No edge moves of T_{n-1} are possible at this point. Therefore, $\textsc{Greedy}$ must make the only remaining edge move, which is $T_n - v_n v_\infty + v_n v_{n-1}$. $\textsc{RevGen}(n, 1, 0)$ also makes this move here, by line 17.

Since $\textsc{Greedy}(L_n)$ and $\textsc{RevGen}(n, 1, 0)$ start with the same tree, produce the same trees in the same order, and halt at the same place, it follows that $\mathcal{R}_n = \textsc{Greedy}(L_n)$. ◀

Because $\textsc{Greedy}(P_n)$ generates unique spanning trees of F_n, Lemma 5 together with Lemma 6 implies the following.

▶ **Lemma 7.** *For $n \geq 2$, $\mathcal{G}_n = \textsc{Greedy}(P_n)$ is a pivot Gray code listing for the spanning trees of F_n.*

It remains to prove how efficiently our pivot Gray codes can be generated. To store the global tree T, the algorithms $\textsc{Gen}$ and $\textsc{RevGen}$ can employ an adjacency list model where each edge uv is associated only with the smallest labeled vertex u or v. This means v_∞ will never have any edges associated with it, and every other vertex will have at most 3 edges in its list. Thus the tree T requires at most $O(n)$ space to store, and edge additions and deletions can be done in constant time. The next result completes the proof of Theorem 4.

▶ **Lemma 8.** *For $n \geq 2$, $\mathcal{G}_n$ and $\mathcal{R}_n$ can be generated in $O(1)$-amortized time using $O(n)$ space.*

Proof. For each call to GEN$(n, s_1, varEdge)$ where $n > 3$, there are at most four recursive function calls, and at least two new spanning trees generated. Thus, the total number of recursive calls made is at most twice the number of spanning trees generated. Each edge addition and deletion can be done in constant time as noted earlier. Thus each recursive call requires a constant amount of work, and hence the overall algorithm will run in $O(1)$-amortized time. There is a constant amount of memory used at each recursive call and the recursive stack goes at most $n-3$ levels deep; this requires $O(n)$ space. As mentioned earlier, the global variable T stored as adjacency lists also requires $O(n)$ space. ◀

3.2 Ranking

Given a spanning tree T in $\mathcal{G}_n$, we calculate its rank by recursively determining which stage (recursive call) T is generated. We can determine the stage by focusing on the presence/absence of the edges v_nv_{n-1}, v_nv_∞, $v_{n-2}v_\infty$, and $v_{n-2}v_{n-1}$. Based on the discussion of the recursive algorithm, there are t_{n-1} trees generated in S1, t_{n-1} trees generated in S2, t_{n-2} trees generated in S3, and $t_{n-2} - t_{n-3}$ trees generated in S4. S3 is partitioned into two cases based on whether $v_{n-2}v_{n-1}$ $(varEdge)$ is present. For the remainder of this section we will let $T_{n-1} = T - v_n$ and $T_{n-2} = T - v_n - v_{n-1}$.

If $v_nv_{n-1}, v_nv_\infty, v_{n-2}v_\infty \in T$, then T is a tree in S4 of $\mathcal{G}_n$. The trees of S4 are the trees of $\mathcal{G}_{n-2}$ without S1, listed in reverse order. So, the rank can be calculated by subtracting the rank of T_{n-2} in $\mathcal{G}_{n-2}$ from $2t_{n-1} + 2t_{n-2} + 1$ (the rank of the last tree of S3 plus t_{n-2}). Note that we do not use $2t_{n-1} + 2t_{n-2} - t_{n-3}$ because the recursive rank calculated already takes into account the trees of S1 that are missing.

If $v_nv_{n-1}, v_nv_\infty \in T$ and $v_{n-2}v_\infty \notin T$, then T is a tree in S3 of $\mathcal{G}_n$. The trees of S3 are the trees of $\mathcal{G}_{n-2}$ where $v_{n-2}v_\infty$ has been replaced by $v_{n-2}v_{n-1}$. So, if $v_{n-2}v_{n-1} \in T$, then in order to recursively calculate the rank of T_{n-2} in $\mathcal{G}_{n-2}$, we need to replace $v_{n-2}v_{n-1}$ with $v_{n-2}v_\infty$. If $v_{n-2}v_{n-1} \notin T$, then no edge replacements are needed. We can then determine the rank of T in $\mathcal{G}_n$ by adding the rank of T_{n-2} in $\mathcal{G}_{n-2}$ to $2t_{n-1}$ (the rank of the last tree of S2).

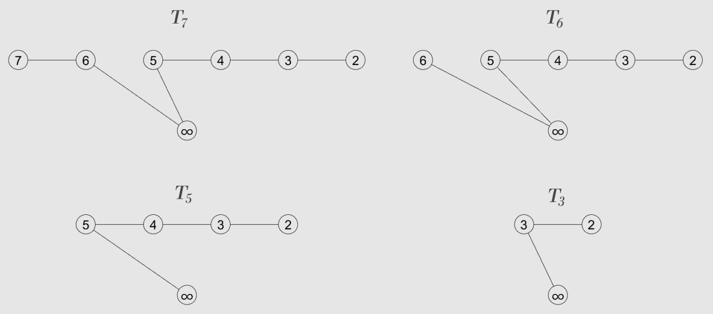
The other two cases are fairly trivial. If $v_nv_{n-1} \in T$ and $v_nv_\infty \notin T$, then T is in S1. Since S1 is the trees of $\mathcal{G}_{n-1}$ with v_nv_{n-1} added, we simply return the ranking of T_{n-1} in $\mathcal{G}_{n-1}$. If $v_nv_\infty \in T$ and $v_nv_{n-1} \notin T$, then T is in S2. Since S2 is the trees of $\mathcal{G}_{n-1}$ in reverse order with v_nv_∞ added, we return $2t_{n-1} + 1$ (the rank of the first tree of S3) minus the rank of T_{n-1} in $\mathcal{G}_{n-1}$.

For $n > 1$, let $R_n(T)$ denote the rank of T in the listing $\mathcal{G}_n$. If $n = 2, 3, 4$, then $R_n(T)$ can easily be derived from Figure 3. Based on the above discussion, for $n \geq 5$:

$$R_n(T) = \begin{cases} 2t_{n-1} + 2t_{n-2} - R_{n-2}(T_{n-2}) + 1 & \text{if } e_1, e_2, e_3 \in T \\ 2t_{n-1} + R_{n-2}(T_{n-2} + e_3) & \text{if } e_1, e_2, e_4 \in T,\ e_3 \notin T \\ 2t_{n-1} + R_{n-2}(T_{n-2}) & \text{if } e_1, e_2 \in T,\ e_3, e_4 \notin T \\ 2t_{n-1} - R_{n-1}(T_{n-1}) + 1 & \text{if } e_2 \in T,\ e_1 \notin T \\ R_{n-1}(T_{n-1}) & \text{if } e_1 \in T,\ e_2 \notin T \end{cases} \tag{1}$$

where $e_1 = v_nv_{n-1}$, $e_2 = v_nv_\infty$, $e_3 = v_{n-2}v_\infty$, and $e_4 = v_{n-2}v_{n-1}$.

Example 3 Consider the spanning trees T_7, T_6, T_5, and T_3 for F_7, F_6, F_5 and F_3 respectively.



Observe that $T_6 = T_7 - v_7$, $T_5 = T_6 - v_6$, and $T_3 = T_5 - v_5 - v_4 + v_3v_\infty$. Consider $R_7(T)$ where $T = T_7$. Applying the formula in (1) we have

$$\begin{aligned}
R_7(T) &= R_6(T_6)\\
&= 2t_5 - R_5(T_5) + 1\\
&= 43 - (2t_4 + R_3(T_3 + v_3v_\infty))\\
&= 43 - (16 + 3)\\
&= 24.
\end{aligned}$$

Since each application of (1) requires constant time, and the recursion goes $O(n)$ levels deep, we arrive at the following result provided the first $2(n-2)$ Fibonacci numbers are precomputed. We note that the calculations are on numbers up to size t_{n-1}.

▶ **Theorem 9.** *The listing $\mathcal{G}_n$ can be ranked in $O(n)$ time using $O(n)$ space under the unit cost RAM model.*

3.3 Unranking

Determining the tree T at rank r, where $0 < r \le t_n$, in the listing $\mathcal{G}_n$ follows similar ideas by constructing T starting from a set of n isolated vertices and adding one edge at a time. If $0 < r \le t_{n-1}$ then T must be a tree in S1 of $\mathcal{G}_n$. So, we can add v_nv_{n-1} to T and consider the rank r tree in $\mathcal{G}_{n-1}$. If $t_{n-1} < r \le 2t_{n-1}$, then T is a tree in S2 of $\mathcal{G}_n$. Since S2 of $\mathcal{G}_n$ is simply $\mathcal{R}_{n-1} + v_nv_\infty$, we can add v_nv_∞ to T and then consider the rank $2t_{n-1} + 1 - r$ (rank of the first tree of S3 minus r) tree in $\mathcal{G}_{n-1}$. If $2t_{n-1} < r \le 2t_{n-1} + t_{n-2}$, then T must be a tree in S3 of $\mathcal{G}_n$. Since all trees in S3 have the edges v_nv_{n-1} and v_nv_∞, we can add these edges to T. Then, we can consider the rank $r - 2t_{n-1}$ (r minus the rank of the last tree of S2) tree of $\mathcal{G}_{n-2}$. Also note that since T is in S3, $v_{n-2}v_{n-1}$ will replace $v_{n-2}v_\infty$ for the trees of $\mathcal{G}_{n-2}$. Otherwise, if $r > 2t_{n-1} + t_{n-2}$, then T must be in S4 of $\mathcal{G}_{n-2}$. Similar to S3, we can add v_nv_{n-1} and v_nv_∞ to T as all trees in S4 have these edges. Then, we consider the $2t_{n-1} + 2t_{n-2} - r + 1$ (rank of the last tree of S3 plus the rank of the last tree of $\mathcal{G}_{n-2}$ minus r) tree of $\mathcal{G}_{n-2}$.

Let $U_n(r, \mathit{replaceEdge})$ return the edges that form the tree T at rank r for the listing $\mathcal{G}_n$. The parameter *replaceEdge* indicates whether or not the edge v_nv_{n+1} should be added instead of v_nv_∞. Initially, r is the specified rank, and $\mathit{replaceEdge} = 0$. In the base cases where $n = 2, 3, 4$, then T is derived from Figure 3. For these cases, if the edge v_nv_∞ is present and *replaceEdge* = 1, then it is replaced by the edge v_nv_{n+1}. Based on the above discussion, we arrive at the following recursive construction for $U_n(r, \mathit{replaceEdge})$.

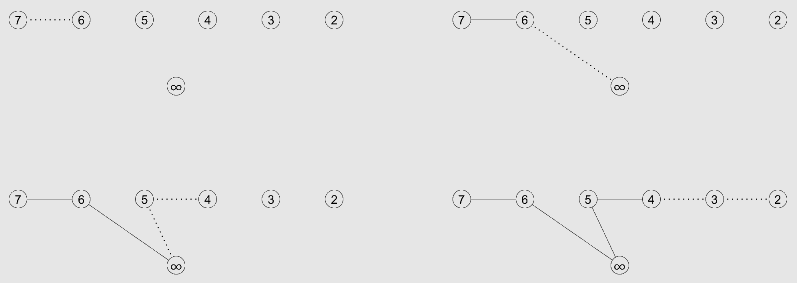
$$U_n(r, \mathit{replaceEdge}) = \begin{cases} U_{n-1}(r,0) + v_nv_{n-1} & \text{if } 0 < r \le t_{n-1}, \\ U_{n-1}(2t_{n-1}-r+1, 0) + e & \text{if } t_{n-1} < r \le 2t_{n-1}, \\ U_{n-2}(r-2t_{n-1}, 1) + v_nv_{n-1} + e & \text{if } 2t_{n-1} < r \le 2t_{n-1}+t_{n-2}, \\ U_{n-2}(2t_{n-1}+2t_{n-2}-r+1, 0) + v_nv_{n-1} + e & \text{otherwise,} \end{cases} \tag{2}$$

where $e = v_nv_{n+1}$ if *replaceEdge* = 1 and $e = v_nv_\infty$ otherwise.

Example 4 To find the 24th tree T in the listing $\mathcal{G}_7$, we consider $U_7(24, 0)$. Repeated application of (2) yields the following

$$\begin{aligned} U_7(24,0) &= U_6(24,0) + v_7v_6 && \triangleright \text{ since } 0 < 24 \le t_6 \\ &= U_5(19,0) + v_7v_6 + v_6v_\infty && \triangleright \text{ since } t_5 < 24 \le 2t_5 \\ &= U_3(3,1) + v_7v_6 + v_6v_\infty + v_5v_4 + v_5v_\infty && \triangleright \text{ since } 2t_4 < 19 \le 2t_4 + t_3 \\ &= v_7v_6 + v_6v_\infty + v_5v_4 + v_5v_\infty + v_3v_3 + v_3v_2 \end{aligned}$$

Reaching a base case, the 3rd tree of $\mathcal{G}_3$ is $\{v_3v_2, v_3v_\infty\}$. Since *replaceEdge* = 1, the edge v_3v_∞ is replaced with v_3v_4 and we end up with the spanning tree T containing the edges from the last line of the equation. These four steps to construct T are illustrated below.



Since each application of (2) requires constant time, and the recursion goes $O(n)$ levels deep, we arrive at the following result provided the first $2(n-2)$ Fibonacci numbers are precomputed. We note that the calculations are on numbers up to size t_{n-1}.

▶ **Theorem 10.** *The listing $\mathcal{G}_n$ can be unranked in $O(n)$ time using $O(n)$ space under the unit cost RAM model.*

4 Conclusion

We answer each of the three Research Questions outlined in Section 1 in the affirmative for the fan graph, F_n. First, we discovered a greedy algorithm that exhaustively listed all spanning trees of F_n experimentally for small n with an easy to define starting tree. We then studied this listings which led to a recursive construction producing the same listing that runs in $O(1)$-amortized time using $O(n)$ space. We also proved that the greedy algorithm does in fact exhaustively list all spanning trees of F_n for all $n \ge 2$, by demonstrating the listing is equivalent to the aforementioned recursive algorithm. It is the first greedy algorithm known to exhaustively list all spanning trees for a non-trivial class of graphs, in a pivot Gray code order. Though, subsequently, a greedy approach for arbitrary graphs can be modified to also produce a pivot Gray code for the fan [27]. Finally, we provided $O(n)$-time ranking and unranking algorithms for our listings, assuming the unit cost RAM model. Some notable open problems that remain:

1. Can other (not necessarily pivot) Gray codes for spanning trees (for interesting classes of graphs) be ranked/unranked in polynomial time?

2. Can a stronger version of **Research Question #1** be answered in the affirmative (even for fan graphs) where we further insist that the pivot Gray code be cyclic, that is, the first and last spanning tree in the listing also differ by the addition of and deletion of a single edge with a common endpoint?

3. Can **Research Question #1** be solved for all graphs (even just the existential part of the question)?

It is expected that in solving the last open question listed above, progress will start with generating pivot Gray codes for other specific families of graphs. To this end, in the upcoming Section 4.1, we adapt our pivot Gray code for the fan to obtain a pivot Gray code for the wheel. It remains an interesting open problem to answer the research questions for other classes of graphs including the n-cube, and complete graph.

4.1 Final comment: The wheel

The wheel W_n is obtained by adding the single edge v_2v_n to F_n. With the addition of this single edge, we were unable to find a greedy algorithm to list all the spanning trees of W_n in a pivot Gray code order. However, we were able to adapt the recursive algorithm for the spanning trees of F_n to obtain a pivot Gray code for W_n by appropriately inserting the spanning trees of W_n that contain the *wheel edge* v_2v_n.

Figure 6 provides an example of a pivot Gray code listing for the spanning trees of W_5 obtained by inserting the trees containing the edge v_5v_2 into the listing for $\mathcal{G}_5$. Note that all the trees containing the wheel edge v_5v_2 contain subgraphs corresponding to the spanning trees of F_4, F_3, or F_2. For example, the second tree in the first row contains the first spanning tree of $\mathcal{G}_4$ as a subgraph (highlighted in red) on the vertices v_4, v_3, v_2, and v_∞. The third tree of the second row also contains the first tree of $\mathcal{G}_4$, except this time it appears as a subgraph (highlighted in blue) on the vertices v_5, v_4, v_3, and v_∞. We now introduce some terminology to differentiate between these two cases. A tree, T, with the wheel edge contains a *right subgraph* if v_2 is connected to v_∞ when the edge v_nv_2 is removed from T. Similarly, T contains a *left subgraph* if v_n is connected to v_∞ when the edge v_nv_2 is removed from T. As a further example of trees that contain right subgraphs, see the third and fourth tree on the first row, which contain the first tree of $\mathcal{G}_3$ and the first tree of $\mathcal{G}_2$, respectively.

When inserting the trees with the wheel edge, it is straightforward to fit in the trees that contain right subgraphs due to the recursive nature of Gen. Since the trees of S1 of Gen$(n, 1, 0)$ are all of the trees of Gen$(n-1, 1, 0)$ with the edge v_nv_{n-1} added, we can add trees containing the wheel edge and these right subgraphs as intermediate trees in between the trees of S1 of $\mathcal{G}_n$ by removing the edge v_nv_{n-1} and adding v_nv_2. We can further insert the trees containing smaller subgraphs (like the third and fourth tree on the first row) by rotating edges along the path, as seen in between f_1 and f_2 of Figure 6. Note that in between the fourth and fifth tree of the first row, we make the original edge move between f_1 and f_2, and then rotate edges back along the path to obtain f_2.

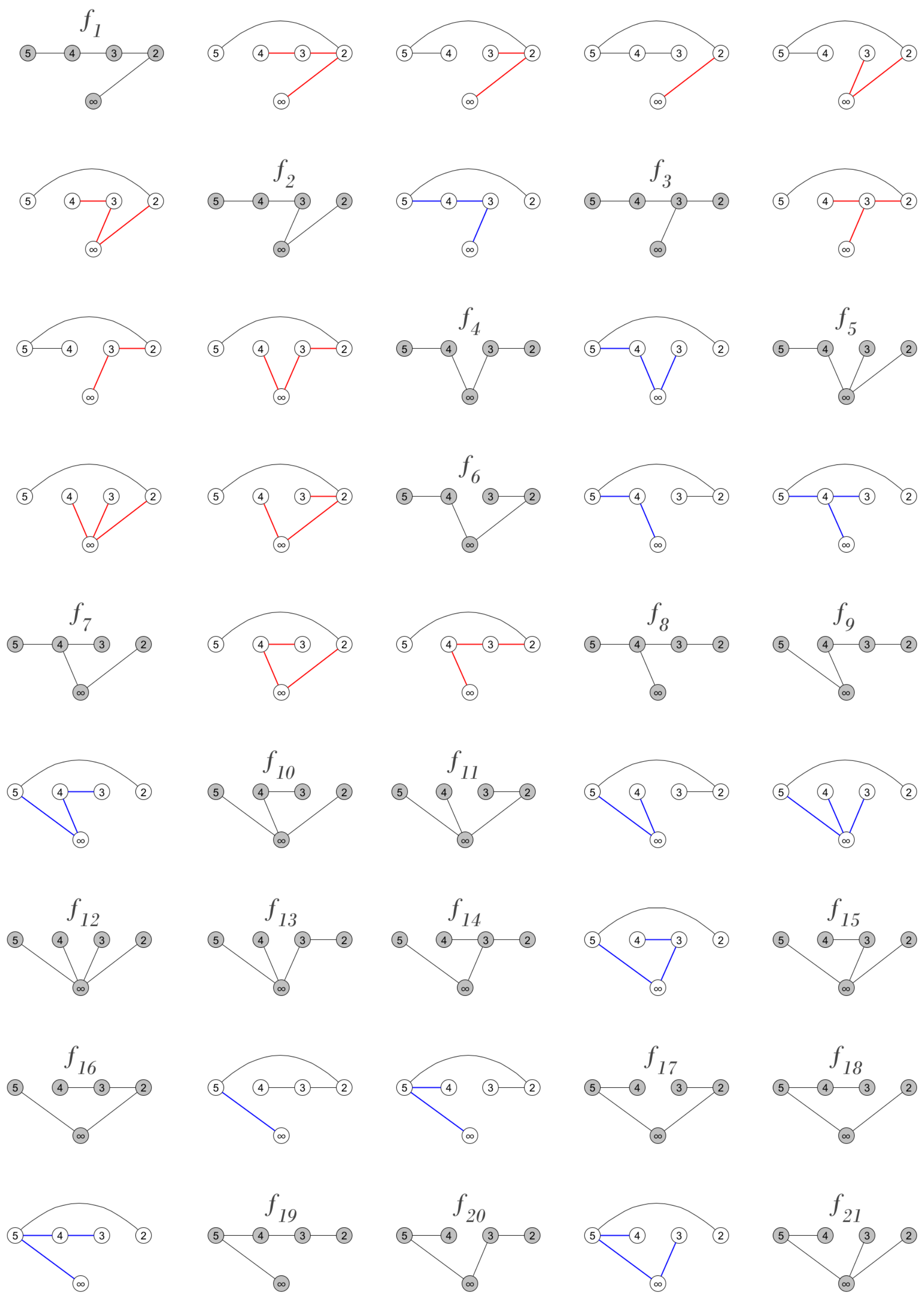


Figure 6 A pivot Gray code listing for the W_5 obtained by inserting the spanning trees with the edge v_2v_5 in between the trees of $\mathcal{G}_5$ (colored with grey vertices). The label f_j denotes the jth tree of $\mathcal{G}_5$. Blue and red are used to highlight the edges of a left or right subgraph, respectively.

Unfortunately, the trees containing left subgraphs do not follow a nice recursive pattern. However, we are still able to insert them appropriately by considering two cases. First, when v_2 is the pivot vertex, we can insert a single tree containing a left subgraph as an intermediate edge move. As an example of this, see the tree in between f_2 and f_3 of Figure 6. The other case is when v_2v_∞ is present and v_2 is not the pivot vertex, in which case we can insert two trees. For example, in between the trees labeled f_6 and f_7 in Figure 6. Note that we require v_2v_∞ to be present so that we can remove it and add v_nv_2 as an intermediate move. If we instead replaced either v_2v_∞ or v_2v_3 with v_nv_2, then we could end up with duplicate trees. Also note that generating right subgraphs takes precedence over generating left subgraphs. For example, even though our first case is satisfied in between f_7 and f_8, there are still trees with right subgraphs that can be generated, so we do not generate a tree with a left subgraph. Finally, note that inserting trees in the ways we have described does not change the relative order that the trees of $\mathcal{G}_n$ appear.

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A C Code

```
#include <stdio.h>
#include <stdlib.h>
#define MAX_N 30

int n;
int tree[MAX_N+2][MAX_N+2];     // Adjacency matrix of a spanning tree
long long int fib[2*MAX_N+1];    // Stores the Fibonacci numbers
long long int numTrees = 1;      // Number of trees generated

void ReverseGen(int k, int S1, int varEdge);

//--------------------------------------------------
void PrintMove(int v, int old, int new) {
    printf("Move #%lld: -(%d, %d) +(%d, %d)\n", numTrees, v-1, old-1, v-1, new-1);
}
//--------------------------------------------------
void PrintTree() { // Prints edge list of tree

  for (int i = 2; i < n+1; i++) {
    for (int j = i+1; j < n+2; j++) {
      if (tree[i][j] == 1) {
        printf("%d %d\n", i-1, j-1);
      }
    }
  } printf("\n");

}
//--------------------------------------------------
int tF(int k) { return fib[2*k - 2]; } // Calculates t(F_k)

//--------------------------------------------------
void CreateStartTree() { // Creates adjacency matrix of P_n
    tree[2][n+1] = tree[n+1][2] = 1;
    for (int i = 3; i < n+1; i++) {
        tree[i][i-1] = tree[i-1][i] = 1;
    }
}
//--------------------------------------------------
void CreateLastTree(int k) { // Creates adjacency matrix of L_n
    if (k == 2) {
        tree[2][n+1] = tree[n+1][2] = 1;
    } else if (k == 3) {
        tree[2][3] = tree[3][2] = 1;
        tree[3][n+1] = tree[n+1][3] = 1;
    } else if (k == 4) {
        tree[2][3] = tree[3][2] = 1;
        tree[3][4] = tree[4][3] = 1;
        tree[4][n+1] = tree[n+1][4] = 1;
    } else if (k > 4){
        tree[k][k-1] = tree[k-1][k] = 1;
        tree[k][n+1] = tree[n+1][k] = 1;
        tree[k-2][n+1] = tree[n+1][k-2] = 1;
        CreateLastTree(k-3);
    }
}
//--------------------------------------------------
void CreateFib() { // Populates the Fibonacci array
    fib[1] = fib[2] = 1;
    for (int i = 3; i <= 2*(MAX_N-1); i++) fib[i] = fib[i-1] + fib[i-2];
}
//--------------------------------------------------
int Rank(int k) {

    // Base cases
    if (k == 3) { // F_3
        if (tree[k][k-1] == 1 && tree[k][n+1] == 1) return 3;
        else if (tree[k][n+1] == 1) return 2;
        else if (tree[k][k-1] == 1) return 1;
    }
    else if (k == 2) { // F_2
        if (tree[k][n+1] == 1) return 1;
    }

    if (tree[k][k-1] == 1 && tree[k][n+1] == 1) { // Both edges present
```

```
        if (tree[k-2][n+1] == 1) return 2*tF(k-1) + 2*tF(k-2) - Rank(k-2) + 1*(k!=4); //  S4
        else if (tree[k-2][k-1] == 1) {

            // Delete e_4, Add e_3 and continue as normal
            tree[k-2][k-1] = tree[k-1][k-2] = 0;
            tree[k-2][n+1] = tree[n+1][k-2] = 1;
        }
        return 2*tF(k-1) + Rank(k-2) + 1*(k==4); // S3
    }
    else if (tree[k][k-1] == 1) return Rank(k-1); // S1
    else if (tree[k][n+1] == 1) return 2*tF(k-1) - Rank(k-1) + 1; // S2
    return 0;
}
//-------------------------------------------------
void Unrank(int k, int rank, int replaceEdge) {

    // Base cases
    if (k == 2) { // F_2
        if (replaceEdge == 1) tree[3][2] = tree[2][3] = 1;
        else tree[n+1][2] = tree[2][n+1] = 1;
        return;
    }
    else if (k == 3) { // F_3
        if (rank == 1) {
            tree[n+1][2] = tree[2][n+1] = 1;   tree[2][3] = tree[3][2] = 1;
        }
        else if (rank == 2) {
            tree[2][n+1] = tree[n+1][2] = 1;
            if (replaceEdge == 1) tree[3][4] = tree[4][3] = 1;
            else tree[n+1][3] = tree[3][n+1] = 1;
        }
        else if (rank == 3) {
            tree[3][2] = tree[2][3] = 1;
            if (replaceEdge == 1) tree[3][4] = tree[4][3] = 1;
            else tree[3][n+1] = tree[n+1][3] = 1;
        }
        return;
    }
    if (rank <= tF(k-1)) { // S1 - Add e_1
        tree[k][k-1] = tree[k-1][k] = 1;
        Unrank(k-1, rank, 0);
    }
    else if (rank <= 2*tF(k-1)) { // S2 - Add e_2
        if (replaceEdge == 1) tree[k][k+1] = tree[k+1][k] = 1;
        else tree[k][n+1] = tree[n+1][k] = 1;
        Unrank(k-1, 2*tF(k-1) - rank + 1, 0);
    }
    else if (rank <= 2*tF(k-1) + tF(k-2)) { // S3 - Add both edges
        tree[k][k-1] = tree[k-1][k] = 1;
        if (replaceEdge == 1) tree[k][k+1] = tree[k+1][k] = 1;
        else tree[k][n+1] = tree[n+1][k] = 1;
        Unrank(k-2, rank - 2*tF(k-1), 1*(k!=4));
    }
    else if (rank <= 3*tF(k-1) - tF(k-2)) { // S4 - Add both edges
        tree[k][k-1] = tree[k-1][k] = 1;
        if (replaceEdge == 1) tree[k][k+1] = tree[k+1][k] = 1;
        else tree[k][n+1] = tree[n+1][k] = 1;
        Unrank(k-2, 2*tF(k-1) + 2*tF(k-2) - rank + 1, 1*(k==4));
    }
}
//-------------------------------------------------
void Replace(int v, int old, int new) { // Delete (v, old), Add (v, new)
    PrintMove(v, old, new);
    tree[v][old] = tree[old][v] = 0;
    tree[v][new] = tree[new][v] = 1; numTrees++;
}
//-------------------------------------------------
void Gen(int k, int S1, int varEdge) {

    if (k == 2) { // F_2 base case
        if (varEdge == 1) Replace(2, n+1, 3);
    }
    else if (k == 3) { // F_3 base case
        if (S1 == 1) {
            if (varEdge == 0) Replace(3, 2, n+1);
            else Replace(3, 2, 4); // S3
        }
        Replace(2, n+1, 3);
    }
    else {
        if (S1 == 1) {
            Gen(k-1, 1, 0);
            if (varEdge == 0) Replace(k, k-1, n+1);
            else Replace(k, k-1, k+1); // S3
        }
        ReverseGen(k-1, 1, 0);
        Replace(k-1, k-2, k);
        Gen(k-2, 1, 1);
        if (k > 4) Replace(k-2, k-1, n+1);
        ReverseGen(k-2, 0, 0);
    }
}
//-------------------------------------------------
void ReverseGen(int k, int S1, int varEdge) {

    if (k == 2) {
        if (varEdge == 1) Replace(2, 3, n+1);
    }
    else if (k == 3) {
        Replace(2, 3, n+1);
        if (S1 == 1) {
            if (varEdge == 0) Replace(3, n+1, 2);
            else Replace(3, 4, 2);
        }
    }
    else {
        Gen(k-2, 0, 0);
        if (k > 4) Replace(k-2, n+1, k-1);
        ReverseGen(k-2, 1, 1);
        Replace(k-1, k, k-2);
        Gen(k-1, 1, 0);
        if (S1 == 1) {
            if (varEdge == 0) Replace(k, n+1, k-1);
            else Replace(k, k+1, k-1);
            ReverseGen(k-1, 1, 0);
        }
    }
}
//-------------------------------------------------
int main() {
  int choice, rank, v1, v2;

  // User input and error checking
  printf(" ###################################################################################");
  printf("####################################################################################\n\n");
  printf(" This program provides functionality to list the spanning trees of the Fan graph");
  printf(" in a pivot Gray code order, rank a tree in the listing, or unrank a tree in the listing.\n");
  printf(" The vertices on the path are labeled 1 to n-1, and the universal vertex is labeled n.\n\n");
  printf(" ###################################################################################");
  printf("####################################################################################\n\n");

  printf("  1. Pivot Gray code generation (GEN)\n");
  printf("  2. Pivot Gray code generation in reverse order of option 1 (REVGEN)\n");
  printf("  3. Rank a tree in the listing generated by option 1\n");
  printf("  4. Unrank a tree in the listing generated by option 1\n");
  printf("  Enter selection: ");  scanf("%d", &choice);

  if (choice < 1 || choice > 4) {
    printf("Error: Invalid choice.\n");
    exit(0);
  }
  printf("Input n: "); scanf("%d", &n);
  if (n > MAX_N) {
    printf("Error: n is too big. Please try n <= 30.\n");
  }
  CreateFib();

  if (choice == 1) { // GEN

    CreateStartTree();
    printf("\n##### GEN ####\n");
    Gen(n, 1, 0);
    printf("Number of spanning trees of F_%d: %lld\n", n, numTrees);

  } else if (choice == 2) {
    CreateLastTree(n);
    printf("\n#### REVGEN ####\n");
    ReverseGen(n, 1, 0);
    printf("Number of spanning trees of F_%d: %lld\n", n, numTrees);

  } else if (choice == 3) { // RANK

    printf("Enter the edges of the spanning tree in format 'v1 v2'. ");
    printf("If you input edge (v1, v2), do not input edge (v2, v1). ");
    printf("Use labels 1 to n-1 for the vertices on the path (from right to left)");
    printf(", and label n for the universal vertex. ");
    printf("Warning: no error checking is done.\n");
    for (int i = 1; i <= n-1; i++) {
      printf("Edge %d: ", i);
      scanf("%d %d", &v1, &v2);
      tree[v1+1][v2+1] = tree[v2+1][v1+1] = 1;
    }
    printf("Rank of inputted tree in listing for GEN is #%d.\n", Rank(n));

  } else if (choice == 4){ // UNRANK

    printf("Enter rank (between 1 and %lld): ", fib[2*(n-1)]);
    scanf("%d", &rank);
    if (rank < 1 || rank > tF(n)) {
      printf("Error: Invalid input.\n"); exit(0);
    }
    Unrank(n, rank, 0);
    printf("\nTree #%d of GEN for F_%d is: \n\n", rank, n);
    PrintTree();
  }

  return 0;
}
```