Dichotomizing k-vertex-critical H-free graphs for H of order four

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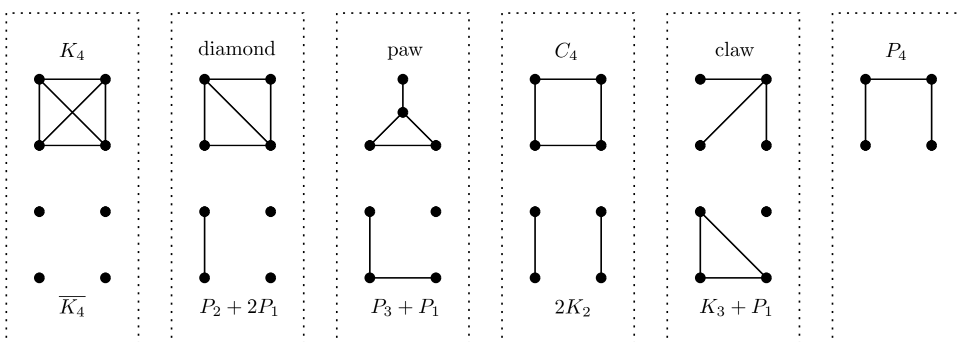
Abstract

For every $k \geq 1$ and $\ell \geq 1$, we prove that there is a finite number of k-vertex-critical $(P_2 + \ell P_1)$-free graphs. This result establishes the existence of new polynomial-time certifying algorithms for deciding the k-colorability of $(P_2 + \ell P_1)$-free graphs. Together with previous research, our result also implies the following characterization: There is a finite number of k-vertex-critical H-free graphs for H of order and for fixed $k \geq 5$ if and only if H is one of $\overline{K_4}, P_4, P_2 + 2P_1$, or $P_3 + P_1$. We also improve the recent known result that there is a finite number of k-vertex-critical $(P_3 + P_1)$-free graphs for all k by showing that such graphs have at most $2k - 1$ vertices. We use this stronger result to exhaustively generate all k-vertex-critical $(P_3 + P_1)$-free graphs for $k \leq 7$.

1 Introduction

There are 11 non-isomorphic simple unlabeled graphs with 4 vertices as illustrated below:



The graphs in each column are complements of each other; P_4 is self-complementary. In this paper we characterize whether or not there is a finite number of k-vertex-critical H-free graphs, where $k \geq 3$ and H has four vertices. The current state of the art, as discussed in Section 1.2, is illustrated in Table 1.

k/H	K_4	$\overline{K_4}$	diamond	$P_2 + 2P_1$	paw	$P_3 + P_1$	$2K_2$	C_4	claw	$K_3 + P_1$	P_4
3	∞	3	∞	2	∞	2	2	∞	∞	∞	1
4	∞	finite	∞	9	∞	8	7	∞	∞	∞	1
5+	∞	finite	∞	?	∞	finite	∞	∞	∞	∞	1

Table 1: The number of k-vertex-critical H-free graphs

The open question on k-vertex-critical $(P_2 + 2P_1)$-free graphs in Table 1 follows as a special case of our main theorem.

Theorem 1.1. *For all $k \geq 1$ and $\ell \geq 1$, there is a finite number of k-vertex-critical $(P_2 + \ell P_1)$-free graphs.*

This leads to the following dichotomy.

Corollary 1.2. *Let H be a graph containing at most four vertices. There is a finite number of k-vertex-critical H-free graphs for fixed $k \geq 5$ if and only if H is an induced subgraph of $\overline{K_4}, P_4, P_2 + 2P_1$, or $P_3 + P_1$.*

While classifying vertex-critical graphs is of interest in its own right, there are also algorithmic consequences that arise from determining that a class of graphs has only finitely many k-vertex-critical graphs. An algorithm is *certifying* if together with each output it also includes a simple and easily verifiable witness that the output is correct. For the problem of k-Coloring, that is, determining if a graph is k-colorable for fixed k, an algorithm that returns "yes" could also include a k-coloring as a certificate. If the algorithm returns "no" to this question, then a certificate could be an induced subgraph that is $(k+1)$-vertex-critical, since any graph that is not k-colorable must contain a $(k+1)$-vertex-critical induced subgraph. If we restrict the input graphs to a class of graphs that contains only finitely many $(k+1)$-vertex-critical graphs, then there is a polynomial-time algorithm to solve k-Coloring that can also return a no-certificate by searching for all $(k+1)$-vertex-critical graphs as induced subgraphs of the input graph (see [1] for more details). This is significant as k-Coloring is NP-complete in general [14]. Moreover, k-Coloring H-free graphs remains NP-complete for all $k \geq 3$ if H contains a cycle [13] or a claw [11, 15]. We note that there are known polynomial-time algorithms to solve k-Coloring for $(P_5 + \ell P_1)$-free [6] and therefore $(P_2 + \ell P_1)$-free graphs, however they are not certifying.

The remainder of this paper is presented as follows. In Section 1.1 we provide essential background definitions. In Section 1.2 we present the necessary background to prove our main result. In Section 2 we prove Theorem 1.1 from which the dichotomizing Corollary 1.2 follows. In Section 3 we present results for k-vertex-critical $(P_3 + P_1)$-free graphs that strengthen known results and allow for the exhaustive generation of these graphs for $k \leq 7$. We conclude with future directions and open problems in Section 4.

1.1 Preliminary background, definitions, and notation

Let $G = (V(G), E(G))$ denote a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $\overline{G}$ denote the complement of G. The *order* of G is number of vertices in G, i.e. $|V(G)|$. Let K_t denote the complete graph on $t \geq 1$ vertices, let C_t denote the induced cycle on $t \geq 3$ vertices, and let P_t denote the induced path on $t \geq 1$ vertices. For graphs $H_1, H_2, \ldots, H_j$, a graph is $(H_1, H_2, \ldots, H_j)$-free if it does not contain H_i as an induced subgraph for any $i = 1, 2, \ldots, j$. If $j = 1$, we simply write H_1-free. A *proper coloring* of a graph is an assignment of colors to the vertices such that adjacent vertices receive different colors. Although there are other notions of graph coloring, in this paper, coloring refers to proper coloring. The disjoint union of two graphs is denoted $G + H$. The graph $\underbrace{G + G + \cdots + G}_{\ell}$ is denoted ℓG. The *join* of two graphs G and H is the graph obtained from $G + H$ by adding all possible edges between G and H and is denoted $G \vee H$. Note that join can be extended to more than two graphs as an associative and commutative operation. A *complete multipartite* graph is one that is isomorphic to $\overline{K_{m_1}} \vee \overline{K_{m_2}} \vee \cdots \vee \overline{K_{m_n}}$ for some $n \geq 1$. For $S \subseteq V(G)$, let $G - S$ denote the graph obtained by removing all vertices in S from G along with all of their incident edges. If $S = \{v\}$, we simply write $G - v$ for $G - \{v\}$. Let $\chi(G), \alpha(G)$, and $\omega(G)$ denote the *chromatic number* (the minimum number of colors required to color the vertices of G), the *independence number* (the order of the largest independent set), and the *clique number* (the order of the largest clique) of G, respectively. A graph G is *k-vertex-critical* if $\chi(G) = k$ but $\chi(G - v) < k$ for all $v \in V$. For other standard definitions and notations we refer the reader to [21].

Let $R(r, s)$, for $r, s \geq 1$, denote the Ramsey numbers which have the following property:

Theorem 1.3 (Ramsey's Theorem [18])**.** *There exists a least positive integer, denoted $R(r, s)$, such that every graph G with at least $R(r, s)$ vertices contains either a clique on r vertices or an independent set on s vertices.*

1.2 Results on k-vertex-critical graphs

The only k-vertex-critical graphs for $k \leq 2$ are the complete graphs, K_1 and K_2. When $k = 3$, the following well-known fact implies that there is an infinite number of 3-vertex-critical graphs.

Fact 1.4. *G is 3-vertex-critical if and only if G is C_{2n+1} for $n \geq 1$.*

Based on this fact, note that $C_{2n+1} \vee K_m$ is $(3 + m)$-vertex-critical for all $n, m \geq 1$. Thus, there is an infinite number of k-vertex-critical graphs for all $k \geq 4$. However, when we restrict the structure of the graphs in question by forbidding certain induced subgraphs, we can be left with finitely many k-vertex-critical graphs for some values of k. A simple example is to forbid P_3. In this case, we are left with the disjoint union of cliques, and hence the only k-vertex-critical P_3-free graph is K_k for all $k \geq 3$. Note that from Fact 1.4, there are finitely many 3-vertex-critical H-free graphs if and only if, for some n, $C_{2\ell+1}$ is H-free for all $\ell > n$.

An old result of Seinche [19] showed that for a P_4-free graph G, we have $\chi(G) = \omega(G)$. Thus, the following fact holds.

Fact 1.5. *The only k-vertex-critical P_4-free graph is K_k, for $k \geq 1$.*

By applying Ramsey's Theorem, we get the following.

Fact 1.6. *There is a finite number of k-vertex-critical $\overline{K_k}$-free graphs, where $k \geq 1$.*

Forbidding small graphs, however, does not always lead to a finite number of k-vertex-critical graphs. For instance, the following three results are known.

Fact 1.7 ([10])**.** *There is an infinite number of k-vertex-critical $2K_2$-free graphs, where $k \geq 5$.*

A classical result of Erdős [8] that for all k, there exist k-chromatic graphs of arbitrarily large girth gives the next fact.

Fact 1.8. *If H contains an induced cycle, then there is an infinite number of k-vertex-critical H-free graphs, where $k \geq 3$.*

An example of an infinite family of k-vertex-critical claw-free graphs is the family of graphs obtained by taking an odd induced cycle with vertices labeled $1, 2, \ldots, 2t+1$ ($t \geq 2$) and substituting a clique of order $k - 2$ for each vertex with an even label.

Fact 1.9. *There is an infinite number of k-vertex-critical claw-free graphs, where $k \geq 3$.*

The following result is known from concurrent research. In Section 3, we provide a more concise proof of this result including a tight upper bound of $2k-1$ for the number of vertices in a k-vertex-critical $(P_3 + P_1)$-free graph.

Fact 1.10 ([3])**.** *There is a finite number of k-vertex-critical $(P_3 + P_1)$-free graphs, where $k \geq 3$.*

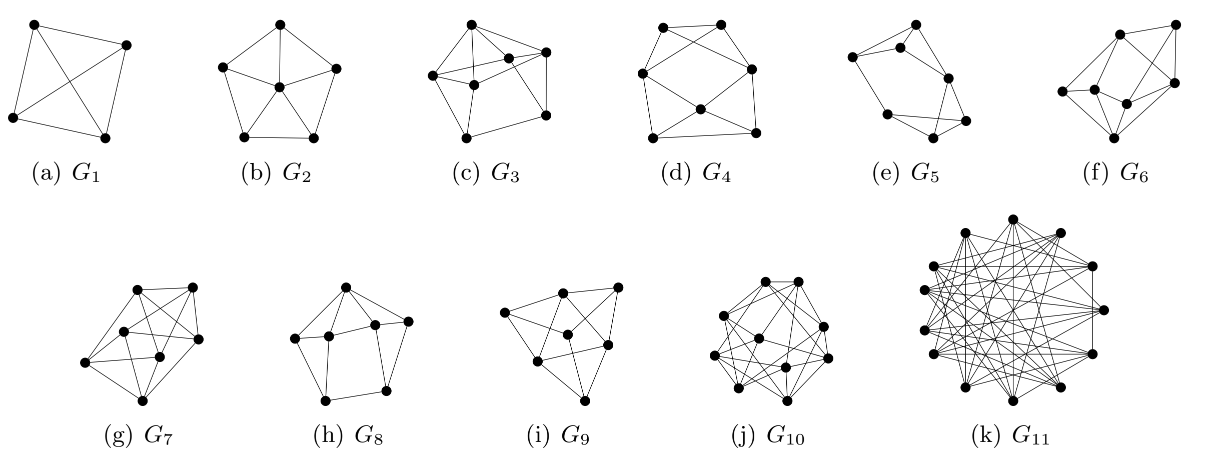
When forbidding two induced subgraphs, there are more positive results with respect to finiteness. For instance, there are exactly 13 5-vertex-critical (P_5, C_5)-free graphs [10]. More generally, there is a finite number of k-vertex-critical $(P_5, \overline{P_5})$-free graphs for all $k \geq 5$ as well as a structural characterization for these graphs [7]. It is also known that there are only finitely many k-vertex-critical $(P_t, K_{s,s})$-free graphs for all k, extending an analogous result for (P_6, C_4)-free graphs [9]. There is a finite number of 4- and 5-vertex-critical (P_6, banner)-free graphs where *banner* is the graph obtained from a C_4 by attaching a leaf to one of its vertices [12]. As well, there are only finitely many 6-vertex-critical (P_5, banner)-free graphs [1]. In recent work, it was shown for $k \geq 5$ that there is a finite number of k-vertex-critical (P_5, H)-free graphs, where H has four vertices and H is neither $2K_2$ nor $K_3 + P_1$ [3]. One of the more significant recent result is the following dichotomy theorem:

Theorem 1.11 ([4])**.** *Let H be a graph. There is a finite number of 4-vertex-critical H-free graphs if and only if H is an induced subgraph of P_6, $2P_3$, or $P_4 + \ell P_1$ for some $\ell \in \mathbb{N}$.*

Applying this result to graphs H on four vertices we obtain the following corollary.

Corollary 1.12. *Let H be a graph containing four vertices. There is a finite number of 4-vertex-critical H-free graphs if and only if H is one of $\overline{K_4}, P_4, P_2 + 2P_1, P_3 + P_1$, or $2K_2$.*

Note that when G is an induced subgraph of H, the set of all G-free graphs is contained in the set of all H-free graphs. Thus, if there are finitely many k-vertex-critical H-free graphs, then there are necessarily finitely many k-vertex-critical G-free graphs. Therefore, except for $\overline{K_4}$, we can determine the number of 4-vertex-critical H-free graphs by filtering from the list of 80 4-vertex-critical P_6-free graphs provided in [5]. In particular, there are nine 4-vertex-critical $(P_2 + 2P_1)$-free graphs, $G_1, G_2, \ldots, G_9$ in Figure 1; there are eight 4-vertex-critical $(P_3 + P_1)$-free graphs, $G_1, G_2, \ldots, G_8$ in Figure 1; there are seven 4-vertex-critical $2K_2$-free graphs, all those from Figure 1 except for G_4, G_5, G_6 and G_8. Using the same list of 80 graphs there are at least 50 4-vertex-critical $\overline{K_4}$-free graphs and the only 4-vertex-critical P_4-free graph is K_4.



(a) G_1 (b) G_2 (c) G_3 (d) G_4 (e) G_5 (f) G_6

(g) G_7 (h) G_8 (i) G_9 (j) G_{10} (k) G_{11}

Figure 1: Examples of 4-vertex-critical graphs.

2 $(P_2 + \ell P_1)$-free graph

In this section we prove Theorem 1.1 and Corollary 1.2. We note that our proof of Theorem 1.1 is inductive and relies on a novel application of Ramsey's Theorem. So, unlike most proofs for showing the finitness of critical graphs, we do not need to appeal to the Strong Perfect Graph Theorem.

If there is a finite number of k-vertex-critical H-free graphs for a given k and H, let $f(k, H)$ denote the maximal number of vertices in such a graph; otherwise, let $f(k, H) = \infty$. Before proving Theorem 1.1, we prove two preliminary results.

Lemma 2.1. *If G is a $(P_2 + \ell P_1)$-free graph and $S \subseteq V(G)$ is a maximal independent set, then every vertex in $G - S$ has at most $\ell - 1$ nonneighbors in S.*

Proof. Consider a vertex $x \in V(G - S)$. Since S is maximal, x has a neighbor in S, say y. If $|S| \leq \ell$, then we are done. Otherwise, suppose x has ℓ nonneighbors in S, say $s_1, s_2, \ldots, s_\ell$. Now, $\{x, y, s_1, s_2, \ldots, s_\ell\}$ induces a $P_2 + \ell P_1$ in G, a contradiction. □

Lemma 2.2. $f(3, P_2 + \ell P_1) = 2\ell + 1$ *for all* $\ell \geq 0$.

Proof. Let G be a 3-vertex-critical $(P_2 + \ell P_1)$-free graph. From Fact 1.4, $G = C_{2m+1}$ for some $m \geq 2$. Moreover, C_{2m+1} is $(P_2 + \ell P_1)$-free if and only if $m \leq \ell$. Therefore, G belongs to the set $\{C_{2m+1} : m = 1, 2, \ldots, \ell\}$. Hence, $f(3, P_2 + \ell P_1) = 2\ell + 1$. □

Proof of Theorem 1.1. The proof is by induction on k. It is clear that $f(1, P_2 + \ell P_1) = 1$, $f(2, P_2 + \ell P_1) = 2$ and the case $k = 3$ was shown in Lemma 2.2. Now let G be a k-vertex-critical $(P_2 + \ell P_1)$-free graph for some $k \geq 4$. Note that since G is k-vertex-critical, $\omega(G) \leq k$. Let S be a maximum independent set of G. If $G - S$ is $(k-2)$-colorable, then we can extend this coloring to a $(k-1)$-coloring of G by coloring all vertices in S the same color, a contradiction. Therefore, $\chi(G - S) = k - 1$. Hence, $G - S$ must have an induced $(k-1)$-vertex-critical subgraph, say G'. Therefore, $|V(G')| \leq f(k-1, P_2 + \ell P_1)$ by the induction hypothesis. Suppose $|S| > (\ell - 1) \cdot f(k-1, P_2 + \ell P_1)$. If there is a vertex $s \in S$ that is adjacent to every vertex in G', then the graph induced by $V(G') \cup \{s\}$ is a proper subgraph of G and is k-vertex-critical, a contradiction. So there are at most $|S|(|V(G') - 1)$ edges between S and G'. On the other hand, by Lemma 2.1, every vertex in G' has at most $\ell - 1$ nonneighbors in S, so the number of edges between G' and S is at least $|V(G')||S| - |V(G')|(\ell - 1)$. If e is the number of edges between G' and S in G, then we have

$$
\begin{aligned}
(\ell - 1) f(k-1, P_2 + \ell P_1) < |S| &\leq |V(G')||S| - e \\
&\leq |V(G')|(\ell - 1) \\
&\leq (\ell - 1) f(k-1, P_2 + \ell P_1),
\end{aligned}
$$

a clear contradiction. Therefore, $|S| \leq (\ell - 1) \cdot f(k-1, P_2 + \ell P_1)$, and $|V(G)| < R(k+1, (\ell - 1) f(k-1, P_2 + \ell P_1) + 1)$, which is finite by Ramsey's Theorem.

Thus, for every ℓ and k, there is an upper bound on the order of every k-vertex-critical $(P_2 + \ell P_1)$-free graph depending only on k and ℓ resulting in only finitely many possibilities for such graphs. □

The dichotomy in Corollary 1.2 is now established by the following points. Let $k \geq 5$.

- ▷ From Theorem 1.1, there is a finite number of k-vertex-critical $(P_2 + 2P_1)$-free graphs.
- ▷ From Fact 1.5, there is exactly one k-vertex-critical P_4-free graph.
- ▷ From Fact 1.6, there is a finite number of k-vertex-critical $\overline{K_4}$-free graphs.

- ▷ From Fact 1.10, there is a finite number of k-vertex-critical $(P_3 + P_1)$-free graphs.
- ▷ From Fact 1.7, there is an infinite number of k-vertex-critical $2K_2$-free graphs.
- ▷ From Fact 1.9, there is an infinite number of k-vertex-critical claw-free graphs.
- ▷ From Fact 1.8, there is an infinite number of k-vertex-critical H-free graphs if H is either K_4, diamond, paw, C_4, or $K_3 + P_1$.

3 $(P_3 + P_1)$-free graphs

In this section we prove the following theorem which improves on Fact 1.10 given in [3]. In particular, we demonstrate that any k-vertex-critical (P_3+P_1)-free graph has at most $2k-1$ vertices. A graph obtaining this bound is $\overline{C_{2k-1}}$.

Theorem 3.1. *Let $k \geq 1$. If G is a k-vertex-critical $(P_3 + P_1)$-free graph, then $\alpha(G) \leq 2$ and $|V(G)| \leq 2k-1$. Moreover, $|V(G)| = 2k-1$ if and only if $\overline{G}$ is connected.*

To prove this theorem, we apply the following known results.

Theorem 3.2 ([17])**.** *A graph G is paw-free if and only if every component of G is triangle-free or complete multipartite.*

Corollary 3.3. *A graph G is (P_3+P_1)-free if and only if $G = H_1 \vee H_2 \vee \cdots \vee H_n$ for some $n \geq 1$ where $\alpha(H_i) \leq 2$ or H_i is the disjoint union of cliques for all $i = 1, 2, \ldots, n$.*

Proof. Since $paw = \overline{P_3 + P_1}$, a graph G is paw-free if and only if $\overline{G}$ is $(P_3 + P_1)$-free. Let G be a paw-free graph. So $G = H_1 + H_2 + \cdots + H_n$ for some $n \geq 1$ where each H_i is triangle-free or a complete multipartite graph from Theorem 3.2. Now, $\overline{H_1 + H_2 + \cdots + H_n} = \overline{H_1} \vee \overline{H_2} \vee \cdots \vee \overline{H_n}$ and $\overline{H_i}$ the disjoint union of cliques or has independence number at most 2 since the complement of a triangle-free graph has independence number at most two and the complement of a complete multipartite graph is the disjoint union of cliques. □

We will need the following two results.

Lemma 3.4 ([7])**.** *A graph $G \vee H$ with $|V(G)|, |V(H)| \geq 1$ is k-vertex-critical if and only if G is k_1-vertex-critical and H is k_2-vertex-critical such that $k_1 + k_2 = k$.*

Theorem 3.5 ([20])**.** *If G is k-vertex-critical and $\overline{G}$ is connected, then $G - v$ has a $(k-1)$-coloring in which every color class contains at least two vertices, for all $v \in V(G)$.*

An immediate consequence of this theorem is that a k-vertex-critical graph that is not the result of a graph join has at least $2k-1$ vertices.

Proof of Theorem 3.1. If $G = F \vee H$ is a k-vertex-critical (P_3+P_1)-free graph, then by Lemma 3.4 and induction on k, $|V(F)| \leq 2k_1 - 1$ and $|V(H)| \leq 2k_2 - 1$ where $k_1 + k_2 = k$ and $\alpha(F), \alpha(H) \leq 2$. Therefore, $|V(G)| \leq 2k-2$ and $\alpha(G) \leq 2$. So the result follows by induction.

Otherwise $\overline{G}$ is connected, so from Corollary 3.3 and the fact that G is vertex-critical and therefore connected, it follows that $\alpha(G) = 2$. Moreover, an elementary lower bound on $\chi(G)$ is $\frac{|V(G)|}{\alpha(G)}$ and since $\chi(G) = k$ and $\alpha(G) = 2$, we have $|V(G)| \leq 2k$. If $|V(G)| = 2k$, then let $v \in V(G)$ and consider $G - v$. Since $|V(G - v)| = 2k - 1$, every $(k-1)$-coloring of $G - v$ will have at least three vertices colored with the same color. But this means $G - v$ and therefore G has an independent set of order at least three, contradicting $\alpha(G) = 2$. Therefore, $|V(G)| \leq 2k-1$. Also, from Theorem 3.5, $|V(G)| \geq 2k-1$, so if such a G exists it must have order equal to $2k-1$. Finally, $\overline{C_{2k-1}}$ is a k-vertex-critical (P_3+P_1)-free graph of order $2k-1$ for all $k \geq 3$. □

Earlier, we demonstrated that there are eight 4-vertex-critical (P_3+P_1)-free graphs. For $k > 4$, the k-vertex-critical (P_3+P_1)-free graphs were not previously known. However, using our results to restrict the structure and order of k-vertex-critical (P_3+P_1)-free graphs we were able to employ `geng` in `nauty` [16] to generate all graphs with order at most 13 and independence number two (complements of those that are triangle-free) and then run an exhaustive computer search on the resulting graphs to find the exact number of k-vertex-critical (P_3+P_1)-free graphs for $k = 5, 6, 7$[1]. Our findings are summarized in Table 2 and the edge sets of all such graphs for $k = 5$ are available in the Appendix. The graphs for $4 \leq k \leq 7$ in graph6 format are available at [2].

n	4-vertex-critical	5-vertex-critical	6-vertex-critical	7-vertex-critical
4	1	0	0	0
5	0	1	0	0
6	1	0	1	0
7	6	1	0	1
8	0	6	1	0
9	0	170	6	1
10	0	0	171	6
11	0	0	17,828	171
12	0	0	0	17,834
13	0	0	0	6,349,629
total	8	178	18,007	6,367,642

Table 2: Number of k-critical (P_3+P_1)-free graphs of order n for $k \leq 7$.

As can be seen in Table 2, the number of 7-vertex-critical (P_3+P_1)-free graphs is quite large, although finite. In practice, implementing a certifying algorithm for k-Coloring (P_3+P_1)-free graphs would require a complete list of all $(k+1)$-vertex-critical graphs. This is now possible for $k \leq 6$. For $k > 6$, though we do not have complete lists, we have been able to impose some structure on these graphs and we can completely describe those that are the result of graph joins in terms of m-vertex-critical graphs for $m < k+1$. However, when the graph has a connected complement, all we know is that it has to have independence number two and order $2(k+1)-1$. It would be interesting to determine a structural characterization of all k-vertex-critical (P_3+P_1)-free graphs with connected complements, i.e. those with order $2k-1$.

4 Open Problems

In this paper we characterize whether or not there is a finite number of k-vertex-critical H-free graphs for $k \geq 5$ and H a graph on four vertices. A natural extension to this research is to consider all 34 non-isomorphic graphs H on five vertices. Of these 34, only four are cycle-free, $2K_2$-free, and claw-free, namely: $\overline{K_5}, P_2+3P_1, P_3+2P_1$, and P_4+P_1. Thus, based on results in Section 1.2

[1]The computation for $k = 7$ took 12 days on a machine with an Intel Core i5-9400T processor and 16GB of RAM.

and Theorem 1.1, the only undetermined graphs H are $P_3 + 2P_1$ and $P_4 + P_1$. This leads to the following two problems.

Problem 1. For which values of $k \geq 5$ is there a finite number of k-vertex-critical $P_3 + 2P_1$-free graphs?

Problem 2. For which values of $k \geq 5$ is there a finite number of k-vertex-critical $P_4 + P_1$-free graphs?

Answering these two problems would lead to a dichotomy result for graphs on five vertices. More generally, for graphs with $n \geq 5$ vertices, a similar analysis shows that the only undetermined graphs are $P_3 + (n-3)P_1$ and $P_4 + (n-4)P_1$. This leads to the following generalizations of the above problems.

Generalized Problem 1. For which values of $k \geq 5$ and $r \geq 2$ is there a finite number of k-vertex-critical $P_3 + rP_1$-free graphs?

Generalized Problem 2. For which values of $k \geq 5$ and $s \geq 1$ is there a finite number of k-vertex-critical $P_4 + sP_1$-free graphs?

We conclude by making the following conjecture, which if proved correct would answer the above open problems.

Conjecture 4.1. Let $k \geq 5$. There is there a finite number of k-vertex-critical H-free graphs if and only if H is an induced subgraph of $P_4 + sP_1$ for some $s \geq 0$.

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Appendix: 5-vertex-critical $(P_3 + P_1)$-free graphs

Below are the edge sets for all 178 5-vertex-critical $(P_3 + P_1)$-free graphs. Vertices are labelled 0 to $n - 1$ and the edges $\{i, j\}$ are denoted ij.

(1) {01, 02, 03, 04, 12, 13, 14, 23, 24, 34}
(2) {02, 03, 05, 06, 13, 14, 15, 16, 24, 25, 26, 35, 36, 45, 46, 56}
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